

Escaping from an Attractor: Importance Sampling and Rest Points I

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Abstract

We discuss importance sampling schemes for the estimation of finite time exit probabilities of small noise diffusions that involve escape from an equilibrium. A factor that complicates the analysis is that rest points are included in the domain of interest. We build importance sampling schemes with provably good performance both pre-asymptotically, i.e., for fixed size of the noise, and asymptotically, i.e., as the size of the noise goes to zero, and that do not degrade as the time horizon gets large. Simulation studies demonstrate the theoretical results.

1 Introduction

This paper considers the use of importance sampling for estimating hitting or exit probabilities for stochastic processes. The process model is a d -dimensional diffusion $X^\varepsilon \doteq \{X^\varepsilon(s), s \in [0, \infty)\}$ satisfying the stochastic differential equation (SDE)

$$dX^\varepsilon(s) = b(X^\varepsilon(s)) ds + \sqrt{\varepsilon} \sigma(X^\varepsilon(s)) dB(s), \quad X^\varepsilon(0) = x, \quad (1.1)$$

where $\varepsilon > 0$ and $B(s)$ is a standard d -dimensional Wiener process. Of particular interest is the case of gradient flows, $b(x) = -DV(x)$, and constant diffusion coefficient, though many aspects of the analysis are more generally applicable. Let $\mathcal{D} \subset \mathbb{R}^d$ be an open set and denote by τ^ε the exit time of $X^\varepsilon(s)$ from \mathcal{D} . We are concerned with the estimation of quantities such as the probability that X^ε leaves \mathcal{D} before some time $T \in (0, \infty)$, or that it exits through a particular subset $O \subset \mathcal{D}$ before T , and related expected values. The principal novel feature of this work is that the initial point is in the neighborhood of an equilibrium point of the noiseless dynamics.

The estimation of such probabilities has several mathematical and computational difficulties. It is related to the estimation of transition probabilities between different metastable states

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within a given time horizon. As is well known, standard Monte Carlo sampling techniques lead to exponentially large relative errors as the noise coefficient ε tends to zero. When rest points are in the domain of interest the situation is even more complicated than usual. This work will focus on this particularly difficult issue.

The performance of unbiased estimators for rare event problems is usually measured by the size of the second moment of the estimator based on a single simulation. For a well designed scheme the ratio of this second moment to the quantity of interest will not grow too rapidly as $\varepsilon \downarrow 0$. One measure is the exponential rate of decay of the second moment. If this rate of decay is exactly twice the decay rate for the probability of interest, then the scheme is called asymptotically efficient (or weakly efficient). The notion of *strong* efficiency requires that the ratio of the second moment to the square of the probability be bounded above uniformly for all small $\varepsilon > 0$. While such performance is certainly desirable, it is not common when dealing with models such as (1.1) that involve state dependent dynamics and complicated geometries. As we describe below, in some sense both these measures are inadequate for the situation considered here.

A theory based on subsolutions to an associated Hamilton-Jacobi-Bellman (HJB) equation has been developed for the design and performance analysis of importance sampling (see for example [6, 7, 8, 5]). In this approach the change of measure (which for reasons made evident later on we will call the control) used in the importance sampling is defined in terms of the gradient of a subsolution, and the performance, as measured by the decay rate of the second moment, is given by the value of the subsolution at the initial location x . This theory is the starting point of our analysis of (1.1), though as mentioned previously the inclusion of rest points will motivate some further developments.

For any particular class of process models and events, an essential step in the application of this approach is the construction of appropriate subsolutions. In this paper we will exploit the fact that the Freidlin-Wentzell quasipotential [10] can be used to construct various subsolutions for certain *time independent* problems related to (1.1). In addition, for particular but important classes of process models (e.g., gradient systems with constant diffusion matrix), the quasipotential and hence these subsolutions take explicit and simple forms. As we discuss in detail later on, these subsolutions also give subsolutions for the *time dependent* problems, and when T is large the value of the subsolution at the starting point (which now includes time $t = 0$ as well as the location) will be close to the maximal value.

It follows that if the final time T is large enough, then existing theory implies that the estimator based on this subsolution should have a nearly optimal decay rate for its second moment. While this is a valid statement, there is an important qualitative difference between problems which include a rest point in the domain of interest and those which do not. The distinction turns not on the decay rate, which behaves as expected in both situations, but rather depends on the pre-exponential terms not captured by the decay rate. When the domain does not contain a rest point one has simultaneously good rates of decay and control over the pre-exponential terms. However, when a rest point is present schemes based only on this time independent subsolution keep the desired decay rate but lose the good control over the pre-exponential terms. The qualitative difference is related to the fact that in the former case a subsolution designed on the basis of the $\varepsilon = 0$ problem can be shown to give useful bounds for the problem with $\varepsilon > 0$, but in the latter case this is no longer true. This qualitative distinction

will be made precise later on, when we construct non-asymptotic bounds on the second moment for the two cases. When $\varepsilon > 0$ is small but not too small, the loss of performance due to the large pre-exponential term can be significant, rendering the associated importance sampling scheme little better than ordinary Monte Carlo. As $\varepsilon \downarrow 0$ the exponential decay rate dominates, and importance sampling once again gives much greater performance than ordinary Monte Carlo. However, the improvement is less than in the case where rest points are not included, and an approach which avoids this loss of performance would certainly be welcome.

One can show that in fact there is no time independent subsolution that will resolve this difficulty, and so one must bring in some form of time dependence. It is not at all clear how one might incorporate time dependence throughout the entire domain $[0, T] \times \mathcal{D}$. An alternative approach, and the one we follow in this paper, is to combine an explicit solution to an approximating time dependent problem in a neighborhood of the rest point with the time independent subsolution obtained via the quasipotential away from the rest point. As we will show, such an approach will maintain the high decay rate while at the same time properly controlling the pre-exponential term. In the neighborhood of the rest point, one can approximate the dynamics of the diffusion process by a Gauss-Markov process, i.e., a process with a constant diffusion matrix and drift that is affine in the state. For these dynamics and appropriate terminal conditions for the localizing problem, the solution to the related PDE can be constructed in terms to the famous linear/quadratic regulator problem from optimal control theory. As a consequence an explicit and nearly optimal scheme for a surrogate problem can be identified in the neighborhood of the rest point, which is then merged with the explicit scheme based on the quasipotential in that part of the domain where it is particularly effective.

In this paper we analyze the difficulties caused by the presence of rest points in a general setting. We describe and theoretically justify a resolution of these difficulties in the case of dimension one, and present computational data for this case. The construction of the localizing problem is more elaborate in dimension greater than one, and will be presented in a companion paper along with the results of numerical experiments in higher dimensions.

The contents of this paper are as follows. In Section 2 we review the relevant large deviation theory and importance sampling. In Section 3 we discuss the effectiveness of time independent subsolutions. In particular, we show that if rest points are not part of the domain of interest then subsolutions lead to both good decay rates and non-asymptotic bounds for the second moment of the corresponding unbiased estimator. However, when rest points are included in the domain of interest, the situation is more complicated and even if the decay rate is good, the prelimit bounds may not be as good as desired. In Section 4, we present a change of measure for the problem with a rest point for a quadratic potential function with provably good pre-asymptotic and asymptotic performance and which does not degrade as T gets larger. In Section 5 we extend the discussion to the nonlinear problem with rest points. Simulation data, demonstrating the discussions in Sections 4 and 5 are also presented in the corresponding sections. The Appendix has some auxiliary lemmas that are used in the main body of the manuscript.

2 Related Large Deviation and Importance Sampling Results

In this section we recall well known large deviation results for probabilities of exit times (Subsection 2.1), review importance sampling in the context of small noise diffusions (Subsection 2.2), and also recall the notion of subsolutions to certain related HJB equations (Subsection 2.3).

In most of this paper the following assumptions will be used. The assumptions are stronger than necessary, but simplify the discussion considerably. For example, the non-degeneracy of the diffusion matrix and regularity of the boundary of \mathcal{D} easily imply that a limit exists for (2.2). They can be weakened, but the existence of the limit then requires conditions that are best addressed in a problem dependent fashion.

Condition 2.1 *i. The drift b is bounded and Lipschitz continuous.*

ii. The coefficient σ is bounded, Lipschitz continuous and uniformly nondegenerate.

iii. \mathcal{D} is an open and bounded subset of \mathbb{R}^d , and at all points on its boundary \mathcal{D} satisfies an interior and exterior cone condition, i.e., there is $\delta > 0$ such that if $x \in \partial\mathcal{D}$, then there exist unit vectors $v_1, v_2 \in \mathbb{R}^d$ such that

$$\{y : \|y - x\| < \delta \text{ and } |\langle y - x, v_1 \rangle| < \delta \|y - x\|\} \subset \mathcal{D}$$

and

$$\{y : \|y - x\| < \delta \text{ and } |\langle y - x, v_2 \rangle| < \delta \|y - x\|\} \cap \mathcal{D} = \emptyset.$$

We also assume that if $x \in \partial\mathcal{D}$ and if ϕ solves $\dot{\phi} = b(\phi)$, $\phi(0) = x$, then $\phi(t) \in \mathcal{D}$ for all $t \in (0, \infty)$.

2.1 Large deviation results

Fix $T \in (0, \infty)$ and consider an initial point $(t, x) \in [0, T) \times \mathcal{D}$. Consider a bounded and class \mathcal{C}^2 function $h : \mathbb{R}^d \mapsto \mathbb{R}$. Let $\mathbb{E}_{t,x}$ denote expected value given $X^\varepsilon(t) = x$, and define

$$\theta^\varepsilon(t, x) \doteq \mathbb{E}_{t,x} \left[e^{-\frac{1}{\varepsilon} h(X^\varepsilon(\tau^\varepsilon))} 1_{\{\tau^\varepsilon \leq T\}} \right]. \quad (2.1)$$

Since θ^ε scales exponentially it is also useful to define

$$G^\varepsilon(t, x) \doteq -\varepsilon \ln \theta^\varepsilon(t, x). \quad (2.2)$$

Although for now we focus on the case where h is bounded and continuous, one is also interested in cases where h is discontinuous and takes the value ∞ . An example is when for some set $O \subset \partial\mathcal{D}$, $h(x) = \infty$ if $x \notin O$ and $h(x) = 0$ if $x \in O$. In this case $\theta^\varepsilon(t, x)$ equals the probability of exiting \mathcal{D} through O before time T . For these cases and under mild regularity conditions on O , statements analogous to Theorem 2.2 below hold.

Let $\mathcal{AC}([t, T] : \mathbb{R}^d)$ be the set of absolutely continuous functions on $[t, T]$ with values in \mathbb{R}^d . We denote the local rate function by

$$L(x, v) \doteq \frac{1}{2} \langle v - b(x), a^{-1}(x) [v - b(x)] \rangle$$

where $a(x) = \sigma(x)\sigma^T(x)$, and the corresponding rate or action functional for $\phi \in \mathcal{AC}([t, T] : \mathbb{R}^d)$ by

$$I_{tT}(\phi) \doteq \int_t^T L(\phi(s), \dot{\phi}(s)) ds.$$

For all other $\phi \in \mathcal{C}([t, T] : \mathbb{R}^d)$ set $I_{tT}(\phi) = \infty$.

The following large deviations result is well known, e.g., [9, 10].

Theorem 2.2 *Assume Condition 2.1. Then for each $(t, x) \in [0, T) \times \mathcal{D}$*

$$\lim_{\varepsilon \downarrow 0} G^\varepsilon(t, x) = G(t, x) \doteq \inf_{\phi \in \Lambda(t, x)} [I_{tT}(\phi) + h(\phi(T))],$$

where

$$\Lambda(t, x) = \left\{ \phi \in \mathcal{C}([t, T] : \mathbb{R}^d) : \phi(t) = x, \phi(s) \in \mathcal{D} \text{ for } s \in [t, T], \phi(T) \in \partial\mathcal{D} \right\}.$$

2.2 Preliminaries on importance sampling

We briefly review the use of importance sampling for estimating $\theta^\varepsilon(t, x)$ for a given function h . Let $\Gamma^\varepsilon(t, x)$ be any unbiased estimator of $\theta^\varepsilon(t, x)$ that is defined on some probability space with probability measure $\bar{\mathbb{P}}$. Thus $\Gamma^\varepsilon(t, x)$ is a random variable such that

$$\bar{\mathbb{E}}\Gamma^\varepsilon(t, x) = \theta^\varepsilon(t, x),$$

where $\bar{\mathbb{E}}$ is the expectation operator associated with $\bar{\mathbb{P}}$. In this paper we will consider only unbiased estimators.

In Monte Carlo simulation, one generates a number of independent copies of $\Gamma^\varepsilon(t, x)$ and the estimate is the sample mean. The specific number of samples required depends on the desired accuracy, which is measured by the variance of the sample mean. However, since the samples are independent it suffices to consider the variance of a single sample. Because of unbiasedness, minimizing the variance is equivalent to minimizing the second moment. By Jensen's inequality

$$\bar{\mathbb{E}}(\Gamma^\varepsilon(t, x))^2 \geq (\bar{\mathbb{E}}\Gamma^\varepsilon(t, x))^2 = \theta^\varepsilon(t, x)^2.$$

It then follows from Theorem 2.2 that

$$\limsup_{\varepsilon \rightarrow 0} -\varepsilon \log \bar{\mathbb{E}}(\Gamma^\varepsilon(t, x))^2 \leq 2G(t, x),$$

and thus $2G(t, x)$ is the best possible rate of decay of the second moment. If

$$\liminf_{\varepsilon \rightarrow 0} -\varepsilon \log \bar{\mathbb{E}}(\Gamma^\varepsilon(t, x))^2 \geq 2G(t, x),$$

then $\Gamma^\varepsilon(t, x)$ achieves this best decay rate, and is said to be *asymptotically optimal*. While asymptotic optimality or near asymptotic optimality is desirable, as noted in the introduction one may also desire good behavior of the pre-exponential term. To keep the terminology clear

we will avoid the conventional usage of terms such as asymptotic optimality, and refer instead to properties of the “decay rate” and the “pre-exponential term.”

The unbiased estimators $\Gamma^\varepsilon(t, x)$ that we consider are all based on measure transformation. Consider $u^\varepsilon(s)$, a sufficiently integrable and adapted function, such that

$$\frac{d\bar{\mathbb{P}}^\varepsilon}{d\mathbb{P}} = \exp \left\{ -\frac{1}{2\varepsilon} \int_t^T \|u^\varepsilon(s)\|^2 ds + \frac{1}{\sqrt{\varepsilon}} \int_t^T \langle u^\varepsilon(s), dB(s) \rangle \right\}$$

defines a family of probability measures $\bar{\mathbb{P}}^\varepsilon$. Then by Girsanov’s Theorem, for each $\varepsilon > 0$

$$\bar{B}(s) = B(s) - \frac{1}{\sqrt{\varepsilon}} \int_t^s u^\varepsilon(\rho) d\rho, \quad t \leq s \leq T$$

is a Brownian motion on $[t, T]$ under the probability measure $\bar{\mathbb{P}}^\varepsilon$, and X^ε satisfies $X^\varepsilon(t) = x$ and

$$dX^\varepsilon(s) = b(X^\varepsilon(s)) ds + \sigma(X^\varepsilon(s)) [\sqrt{\varepsilon} d\bar{B}(s) + u^\varepsilon(s) ds].$$

For our purposes, $u^\varepsilon(s)$ is either given as a process that is progressively measurable with respect to a suitable filtration that measures the Wiener process (sometimes called an *open loop* control), or else it is of *feedback* form, in which case there is a suitably measurable function $\bar{u}^\varepsilon : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $u^\varepsilon(s) = \bar{u}^\varepsilon(s, X^\varepsilon(s))$. Of course when implementing importance sampling we consider only the latter form. Letting

$$\Gamma^\varepsilon(t, x) = \exp \left\{ -\frac{1}{\varepsilon} h(X^\varepsilon(\tau^\varepsilon)) \right\} 1_{\{\tau^\varepsilon \leq T\}} \frac{d\mathbb{P}}{d\bar{\mathbb{P}}^\varepsilon}(X^\varepsilon),$$

it follows easily that under $\bar{\mathbb{P}}^\varepsilon$, $\Gamma^\varepsilon(t, x)$ is an unbiased estimator for $\theta^\varepsilon(t, x)$. The performance of this estimator is characterized by its second moment

$$Q^\varepsilon(t, x; \bar{u}^\varepsilon) \doteq \bar{\mathbb{E}}^\varepsilon \left[\exp \left\{ -\frac{2}{\varepsilon} h(X^\varepsilon(\tau^\varepsilon)) \right\} 1_{\{\tau^\varepsilon \leq T\}} \left(\frac{d\mathbb{P}}{d\bar{\mathbb{P}}^\varepsilon}(X^\varepsilon) \right)^2 \right].$$

The goal of this paper is to investigate the effect of rest points on $Q^\varepsilon(t, x; \bar{u}^\varepsilon)$ and how one can choose controls that guarantee both good decay rates and pre-exponential bounds for $Q^\varepsilon(t, x; \bar{u}^\varepsilon)$.

We conclude this section with a review of subsolutions to related HJB equations. Such subsolutions are essential for constructing and analyzing good important sampling schemes.

2.3 Subsolutions to a related PDE

Let

$$\mathbb{H}(x, p) = \langle b(x), p \rangle - \frac{1}{2} \|\sigma^T(x)p\|^2.$$

The construction of good importance sampling schemes for a quantity such as (2.1) is closely related to the HJB equation

$$U_t(t, x) + \mathbb{H}(x, DU(t, x)) = 0 \text{ for } (t, x) \in [0, T] \times \mathcal{D}, \quad (2.3)$$

$$U(t, x) = h(x) \text{ for } t \leq T, x \in \partial\mathcal{D}, \quad U(T, x) = \infty \text{ for } x \in \mathcal{D}, \quad (2.4)$$

and more precisely to its subsolutions. It can be shown that G defined in Theorem 2.2 is the unique continuous viscosity solution of (2.3) and (2.4), see [9].

Definition 2.3 A function $\bar{U}(t, x) : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is a classical subsolution to the HJB equation (2.3) and (2.4) if

- i. \bar{U} is continuously differentiable,
- ii. $\bar{U}_t(t, x) + \mathbb{H}(x, D\bar{U}(t, x)) \geq 0$ for every $(t, x) \in [0, T] \times \mathcal{D}$,
- iii. $\bar{U}(t, x) \leq h(x)$ for $t \leq T, x \in \partial\mathcal{D}$ and $\bar{U}(T, x) \leq \infty$ for $x \in \mathcal{D}$.

The connection between subsolutions and the performance of importance sampling schemes has been established in several papers, such as [7, 5]. These papers either consider classical subsolutions or, more generally, piecewise classical subsolutions. To simplify the discussion, we consider here just classical subsolutions. In the present setting, we have the following theorem regarding asymptotic optimality (Theorem 4.1 in [5]).

Theorem 2.4 Let $\{X^\varepsilon, \varepsilon > 0\}$ be the unique strong solution to (1.1). Consider a bounded and continuous function $h : \mathbb{R}^d \mapsto \mathbb{R}$ and assume Condition 2.1. Let $\bar{U}(t, x)$ be a subsolution according to Definition 2.3 and define the control $u^\varepsilon(s) = -\sigma^T(X^\varepsilon(s))D\bar{U}(s, X^\varepsilon(s))$. Then

$$\liminf_{\varepsilon \rightarrow 0} -\varepsilon \log Q^\varepsilon(t, x; u^\varepsilon) \geq G(t, x) + \bar{U}(t, x).$$

Since \bar{U} is a subsolution it is automatic that $G(s, y) \geq \bar{U}(s, y)$ for all $(s, y) \in [0, T] \times \mathcal{D}$. If $G(t, x) = \bar{U}(t, x)$ then the scheme has the largest possible decay rate.

3 Qualitative properties of schemes based on subsolutions

In this section we justify some of the claims made in the introduction regarding the differences in performance between importance sampling schemes when rest points are included in the domain of interest and when they are not. We consider just the problem of estimating the probability of escape from a set before time T , and even then consider a particular setup. However, the example will illustrate the difference between the two cases, and also suggest how one might improve the performance of importance sampling when rest points are involved.

Remark 3.1 Much of the prior application of subsolutions to importance sampling [2, 3, 4, 8] has involved the estimation of escape probabilities for classes of stochastic networks, in which case the origin is often the unique stable point for the law of large numbers dynamics [the analogue of (1.1) with $\varepsilon = 0$]. The event most often studied in this context is that of escape from a set (i.e., buffer overflow) before reaching the origin, after starting near but not at the origin. The analogous event for the diffusion model (1.1) is one of the problems that are the focus of the present work. However, the difficulties that will be described momentarily for the diffusion model do not arise when dealing with the analogous estimation problem for stochastic networks, and indeed in that setting the proximity of the rest point has little impact on either the rate of decay or the pre-exponential term. This is related to the fact that the law of large numbers trajectories for stochastic networks reach the origin in finite time, as opposed to the infinite time it takes for the solution to (1.1) with $\varepsilon = 0$ to reach a stable equilibrium point when

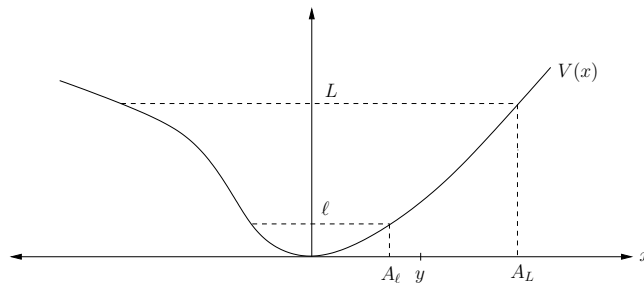


Figure 1: Escape problem with no rest point

not starting at such a point. In turn, this property is responsible for the fact that minimizing trajectories in the definition of the quasipotential are achieved on bounded time intervals for stochastic network models, but take infinite time for processes such as (1.1).

For the remainder of this section we concentrate on the special case of $b(x) = -DV(x)$ and $\sigma(x) = I$, and on a particular estimation problem. We first argue that if the domain of interest does not include a rest point, then given a time-independent subsolution and associated control not only is a good decay rate obtained, but good bounds on the pre-exponential terms hold as well. We then show why this is not possible when a rest point is included.

Assume that $x = 0$ is the global minimum for $V(x)$, so that $DV(0) = 0$, and that $DV(x) \neq 0$ for all $x \neq 0$. Without loss we assume that $V(0) = 0$. Let $0 < \ell < L$ and define $\mathcal{D} = \{x \in \mathbb{R}^d : \ell < V(x) < L\}$ and $A_c = \{x \in \mathbb{R}^d : V(x) = c\}$. Then the problem is to estimate

$$\theta^\varepsilon(t, y) = \mathbb{P}_{t,y} \{X^\varepsilon \text{ hits } A_L \text{ before hitting } A_\ell \text{ and before time } T\},$$

where the initial point y is such that $\ell < V(y) < L$. This corresponds to (2.1), but here h is not bounded and smooth, and instead $h(x) = 0$ if $x \in A_L$ and $h(x) = \infty$ if $x \in A_\ell$. For this problem one can also identify the rate of decay $G(t, y)$ via (2.2). A one-dimensional example is illustrated in Figure 1.

The quasipotential with respect to the equilibrium point 0 is defined by

$$S(0, x) \doteq \inf \{I_{0T}(\phi) : \phi \in \mathcal{C}([0, T] : \mathbb{R}^d), \phi(0) = 0, \phi(T) = x, T \in (0, \infty)\}.$$

It follows from the variational characterization of S that $x \rightarrow S(0, x)$ is always a weak sense solution to $\mathbb{H}(x, -DS(0, x)) = 0$, and therefore by adding an appropriate constant C to satisfy any needed boundary and terminal conditions, $-S(0, x) + C$ will always define a weak sense subsolution. In the present case $S(0, x)$ takes the explicit form (Theorem 4.3.1 in [10])

$$S(0, x) = 2(V(x) - V(0)) = 2V(x),$$

and it is easy to check that $U(x) = -2(V(x) - L)$ is a subsolution according to Definition 2.3. Indeed, $U(x) = 0$ for $x \in A_L$, while the boundary condition $U(x) \leq \infty$ for $x \in A_\ell$ and terminal condition $U(x) \leq \infty$ for $x \in \mathcal{D}$ hold vacuously. See Figure 2. The control (i.e., change of measure) suggested by this subsolution for the importance sampling scheme is $\bar{u}(x) = 2DV(x)$.

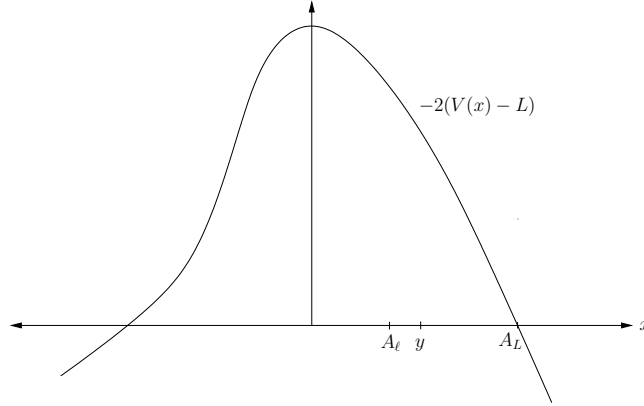


Figure 2: Subsolutions for both cases

The following representation for $Q^\varepsilon(t, x; \bar{u})$ follows essentially from the arguments of Subsection 2.3 of [5], which is based on the representation for exponential integrals with respect to Brownian motion given in [1]. The representation is given in terms of the value of a stochastic differential game, where the player corresponding to the importance sampling scheme has already selected their control (i.e., \bar{u}). The characterization of performance for importance sampling in terms of games was first introduced in [6]. The only difference between the use here and in [5] is that there the function h is bounded, which is not true here since we consider an escape probability. However, the bounds stated below can be obtained by first replacing $\infty 1_{\{\tau > T\}}$ by $M 1_{\{\tau > T\}}$ and then letting $M \uparrow \infty$.

Let \mathfrak{F}_t be a filtration satisfying the usual conditions of completion and right continuity and which measures the Wiener process. Let \mathcal{A} denote the set of all \mathfrak{F}_t -progressively measurable d -dimensional processes $v = \{v(s), 0 \leq s \leq T\}$ that satisfy

$$\mathbb{E} \int_0^T \|v(t)\|^2 dt < \infty.$$

Let $\varepsilon > 0$ be fixed and let \hat{X}^ε be the unique strong solution to

$$d\hat{X}^\varepsilon(s) = -DV(\hat{X}^\varepsilon(s))ds + \left[\sqrt{\varepsilon} dB(s) - [\bar{u}(\hat{X}^\varepsilon(s)) - v(s)]ds \right]$$

with initial condition $\hat{X}^\varepsilon(0) = y$. Let $\hat{\tau}^\varepsilon$ denote the first time \hat{X}^ε exits \mathcal{D} . Then

$$-\varepsilon \log Q^\varepsilon(0, y; \bar{u}) = \inf_{v \in \mathcal{A}} \mathbb{E} \left[\frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} \|v(s)\|^2 ds - \int_0^{\hat{\tau}^\varepsilon} \left\| \bar{u}(\hat{X}^\varepsilon(s)) \right\|^2 ds + \infty 1_{\{\hat{\tau}^\varepsilon > T\}} \right]. \quad (3.1)$$

It is important to note that (3.1) provides a non-asymptotic representation for the performance measure $Q^\varepsilon(0, y; \bar{u})$. However, to obtain a more concrete statement regarding the performance of the importance sampling scheme, we will want bounds on $Q^\varepsilon(0, y; \bar{u})$ that are more explicit than the right hand side of (3.1). We do this by observing that when viewed as a

function of an arbitrary starting point (t, x) , $-\varepsilon \log Q^\varepsilon(t, x; \bar{u})$ also satisfies a nonlinear PDE of the same general form as (2.3) [plus terminal and boundary conditions], and thus lower bounds can be obtained by constructing subsolutions for this PDE. However, a key difference is that in contrast to (2.3), the PDE for (3.1) involves a second derivative term. One cannot avoid this issue, in that second derivative information and ε dependence are needed if one is to obtain non-asymptotic bounds, even when the change of measure is based on a first order equation.

We next give the statement of the lower bound as it applies to the special case of this section. A more general statement and the proof will be given in Lemma 6.1. However, the proof is an easy consequence of Itô's formula and the min/max representation

$$\mathbb{H}(x, p) = \inf_v \sup_u \left[\langle p, -DV(x) - u + v \rangle - \frac{1}{2} \|u\|^2 + \frac{1}{4} \|v\|^2 \right].$$

Define

$$\mathcal{G}^\varepsilon[W](t, x) = W_t(t, x) + \mathbb{H}(x, DW(t, x)) + \frac{\varepsilon}{2} D^2 W(t, x),$$

and let \bar{W} be a subsolution to $\mathcal{G}^\varepsilon[W] = 0$ together with the boundary conditions $W(t, x) = 0$ for $t < T, x \in A_L$, $W(t, x) = \infty$ for $t < T, x \in A_\ell$, and terminal condition $W(T, x) = \infty$ for $x \in \mathcal{D}$. Suppose \bar{u} is the control based on a given smooth function \bar{U} , i.e., $\bar{u}(t, x) = -D\bar{U}(t, x)$. Then

$$\begin{aligned} -\varepsilon \log Q^\varepsilon(0, y; \bar{u}) &= \inf_{v \in \mathcal{A}: \hat{\tau}^\varepsilon \leq T} \mathbb{E} \left[\frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} \|v(s)\|^2 ds - \int_0^{\hat{\tau}^\varepsilon} \left\| \bar{u}(s, \hat{X}^\varepsilon(s)) \right\|^2 ds \right] \\ &\geq 2\bar{W}(0, y) \\ &\quad + \inf_{v \in \mathcal{A}: \hat{\tau}^\varepsilon \leq T} \mathbb{E} \left[\int_0^{\hat{\tau}^\varepsilon} 2\mathcal{G}^\varepsilon[\bar{W}](s, \hat{X}^\varepsilon(s)) ds - \int_0^{\hat{\tau}^\varepsilon} \left\| D\bar{W}(s, \hat{X}^\varepsilon(s)) - D\bar{U}(s, \hat{X}^\varepsilon(s)) \right\|^2 ds \right]. \end{aligned} \tag{3.2}$$

Next we show how (3.2) can be used to obtain bounds that are uniform in T . For $\eta \in (0, 1)$ define

$$U^\eta(x) \doteq (1 - \eta)U(x),$$

where U is the subsolution based on the quasipotential for V as above, and assume that U is twice continuously differentiable. Then as with U the appropriate boundary and terminal inequalities hold for U^η . We next evaluate the right side of (3.2) when $\bar{W} = U^\eta$ and $\bar{U} = U$. A straightforward calculation gives

$$\mathbb{H}(x, DU^\eta(x)) = 2(\eta - \eta^2) \|DV(x)\|^2,$$

and therefore

$$\mathcal{G}^\varepsilon[U^\eta](x) - \frac{1}{2} \|DU^\eta(x) - DU(x)\|^2 = 2(\eta - 2\eta^2) \|DV(x)\|^2 - \varepsilon(1 - \eta) D^2 V(x).$$

For $\varepsilon > 0$ but smaller than a constant that depends on $\inf_{x \in \mathcal{D}} \|DV(x)\|^2$ and $\sup_{x \in \mathcal{D}} \|D^2 V(x)\|^2$, there is $\eta = \eta(\varepsilon)$ with $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that the last display is non-negative. We then obtain from (3.2) the non-asymptotic upper bound

$$Q^\varepsilon(0, y; \bar{u}) \leq e^{-\frac{2}{\varepsilon} U^\eta(y)} = e^{-\frac{2}{\varepsilon} (1 - \eta) U(y)}.$$

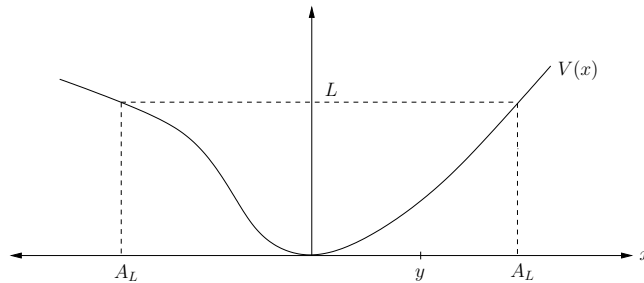


Figure 3: Escape problem with a rest point

Note that this bound is independent of T , and also that this argument is not possible when $0 \in \mathcal{D}$. Indeed, since $\mathbb{H}(0, p) = -\|p\|^2/2 \leq 0$ for all p , $\mathcal{G}^\varepsilon[U^\eta](x) \geq 0$ is not possible for any choice of $\eta < 1$ when $0 \in \mathcal{D}$.

The quality of the bound obtained by this method depends on the degree to which the subsolution obtained for the PDE $\mathcal{G}^\varepsilon[W] = 0$ (plus boundary and terminal conditions) accurately approximates the solution to this equation. In this example, we have used quite crude method to produce such a subsolution, which is to simply reduce a given subsolution to the $\varepsilon = 0$ equation by a constant factor of $(1 - \eta)$. An examination of the calculations suggest that the bound is not at all tight, which turns out to be true. In fact, in this situation we can construct a better subsolution and hence a tighter bound. For example, so long as the origin is not included $x^2 - L + \varepsilon \log(x/\sqrt{L})$ can be used to obtain tighter bounds, though this function is not convenient for the time dependent problem.

Note that the two functions U and U_η play very different roles here. One is used to design an importance sampling scheme [here U], and one used for its analysis [here U_η]. Indeed, U_η with $\eta > 0$ is used only for the analysis of the scheme that corresponds to U and in particular to derive a bound that is independent of T . However, the design of the scheme and thus the simulation algorithm use the control $\bar{u}(t, x) = -D\bar{U}(t, x)$.

Next we consider the behavior of $Q^\varepsilon(0, y; \bar{u})$ when $0 \in \mathcal{D}$. In this case, we claim that $Q^\varepsilon(0, y; \bar{u})$ grows without bound in T for all $\varepsilon > 0$, and therefore the performance of the control based on the quasipotential degrades as T becomes large. To show this is true, we use the game representation to establish a lower bound on $Q^\varepsilon(0, y; \bar{u})$. We again examine a particular situation, which is to estimate the probability of escape from $\mathcal{D} = \{x \in \mathbb{R}^d : V(x) < L\}$ before time T , after starting at y at time 0. See Figure 3. The subsolution is still that of Figure 2.

With the understanding that $\hat{\tau}^\varepsilon$ now represents the time of escape of \hat{X}^ε from $\{x \in \mathbb{R}^d : V(x) < L\}$, the representation (3.1) is still valid. Suppose that T is large and note that, while $\bar{u}(x) = -DU(x) = 2DV(x)$ destabilizes the origin when used to construct the measure used for importance sampling, in the representation (3.1) it actually *increases* the stability of the origin, in the sense that $-DV(x) - \bar{u}(x) = -3DV(x)$. As a consequence, it is easy to construct a control v which shows poor performance as $T \rightarrow \infty$. The construction is suggested in Figure 4. With T large we divide $[0, T]$ into an initial part $[0, T - K)$ and a final part $[T - K, T]$, with K fixed. During the first part we apply $v(t) = 0$. Because the resulting dynamics of \hat{X}^ε are stable

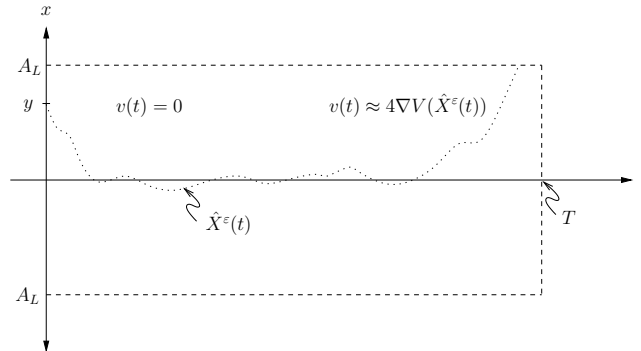


Figure 4: Construction of v

about the origin, with very high probability the process settles around 0 for the entire interval $[0, T - K)$. In the game representation there is then a running cost of $\|\bar{u}(\hat{X}^\varepsilon(s))\|^2$, which one can check is of order $\varepsilon > 0$. In the second portion we apply a control which leads to escape prior to time T , with a cost that may depend on K but is independent of T . An example of such a control, at least away from the origin, is as illustrated in Figure 4. The precise details of the construction in this second part are not important. All that is needed is that such a control exists, which can easily be demonstrated.

When the two parts are combined, we have a control that provides an upper bound of the form

$$-\varepsilon \log Q^\varepsilon(0, y; \bar{u}) \leq -\varepsilon C_1[T - K] + C_2,$$

where C_1 and C_2 are positive constants. This shows that

$$Q^\varepsilon(0, y; \bar{u}) \geq e^{C_1[T-K]} e^{-\frac{1}{\varepsilon} C_2}, \quad (3.3)$$

and thus for fixed $\varepsilon > 0$ and large T the term we have called the pre-exponential term dominates and the scheme is very far from optimal. We find that in this situation there are two exponential scalings, one in the noise strength and one in the length of the time interval, and the issue of which dominates depends on their relative sizes.

These effects are reflected in computational data. In Tables 1 and 2 we present both estimated values and relative errors for the problem of escape from an interval of the form $[-A, A]$, with $A = 1$. The process is a one dimensional Gauss-Markov model with drift $-cx$ and diffusion coefficient $\sqrt{\varepsilon}\sigma$ [see (4.1)], with $c = \sigma = 1$. In the tables, values of T appear at the top and values of ε along the left side. Each computed value is based on 10^7 samples. A dash indicates that no samples escaped. To ease the presentation the relative errors are rounded to the nearest integer. Owing to the fact that the subsolution based on the quasipotential is far from optimal in any sense when T is small, the relative errors are large for small T and decrease until approximately $T = 2$. For larger T the errors grow rapidly with T . Note that the estimated relative errors in Table 2 are not necessarily accurate for large T , since they are subject to the same errors that can affect the probability estimates, but do indicate the qualitative worsening of estimation accuracy.

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	10	14	18
0.20	$9.8e-07$	$2.0e-04$	$3.3e-03$	$9.0e-03$	$9.1e-03$	$1.2e-01$	$1.7e-01$	$2.0e-01$
0.16	$3.6e-08$	$2.5e-05$	$7.3e-04$	$2.2e-03$	$6.6e-03$	$4.0e-02$	$5.7e-02$	$6.4e-02$
0.13	$9.8e-10$	$2.3e-06$	$1.3e-04$	$5.1e-04$	$1.6e-03$	$1.1e-02$	$1.5e-02$	$2.7e-02$
0.11	$4.7e-11$	$2.4e-07$	$2.4e-05$	$1.1e-04$	$3.9e-04$	$2.8e-03$	$4.0e-03$	$6.0e-03$
0.09	—	$8.8e-09$	$2.1e-06$	$1.2e-05$	$5.2e-05$	$4.0e-04$	$5.8e-04$	$8.3e-04$
0.07	—	$5.5e-11$	$5.0e-08$	$4.3e-07$	$2.2e-06$	$1.9e-05$	$2.8e-05$	$3.7e-05$
0.05	—	$5.6e-15$	$5.9e-11$	$9.7e-10$	$6.9e-09$	$7.1e-08$	$1.1e-07$	$1.3e-07$

Table 1: Using the subsolution based on quasipotential throughout. Estimated values for different pairs (ε, T) , two sided problem.

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	10	14	18
0.20	91	7	2	1	1	10	51	179
0.16	253	10	2	1	1	10	48	139
0.13	748	16	3	1	1	9	48	378
0.11	1594	26	3	1	1	10	42	272
0.09	—	49	4	2	1	9	43	357
0.07	—	127	5	2	1	8	47	251
0.05	—	714	8	2	1	8	42	145

Table 2: Using the subsolution based on quasipotential throughout. Relative errors per sample for different pairs (ε, T) , two sided problem.

Tables 3 and 4 present the approximated values and relative errors for the problem with the domain $(-\infty, A]$ and escape possible therefore only at A . The results are of the same qualitative form as before, and carried out only to $T = 10$.

It is useful to compare the two situations, and identify why uniform control of pre-exponential terms was not possible when $0 \in \mathcal{D}$. In both cases the control was based on the quasipotential, which is a valid subsolution to the $\varepsilon = 0$ problem. When using Ito's formula to bound the second moment of the estimator, we must of course deal with the second derivative term, which is multiplied by $\varepsilon > 0$. It can happen that this term has a sign that degrades (increases) the second moment, and indeed this is always true in a neighborhood of the origin (this is essentially due to the convexity of V near the origin). For the case where $0 \notin \mathcal{D}$ and for sufficiently small $\varepsilon > 0$, this could be balanced by using that when $x \neq 0$ $\bar{u}(x)$ and therefore $D\bar{U}(x)$ are nonzero. However, this is not possible when the rest point is included in the domain of interest. Indeed, the running cost that is accumulated in the construction leading to (3.3) corresponds to this term, and as that argument shows it cannot be removed. The construction also suggests how the large variance comes about, which is that some trajectories generated under the change of measure defined by \bar{u} remain in a neighborhood of the origin for a long time, in spite of the fact that with such dynamics the origin is an unstable equilibrium point. The likelihood ratios along these trajectories can vary greatly and, even though they are themselves relatively unlikely, they are likely enough to increase the variance of the estimator to the point where it will become worse than standard Monte Carlo. As such, they are reminiscent of the “rogue” trajectories

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	7	10
0.20	$4.6e-07$	$1.0e-04$	$1.7e-03$	$4.5e-03$	$1.1e-02$	$4.2e-02$	$6.2e-02$
0.16	$2.1e-08$	$1.3e-05$	$3.7e-04$	$1.2e-03$	$3.3e-03$	$1.3e-02$	$2.0e-02$
0.13	$2.7e-10$	$1.1e-06$	$6.5e-05$	$2.5e-04$	$7.9e-04$	$3.5e-03$	$5.3e-03$
0.11	$1.4e-11$	$1.2e-07$	$1.2e-05$	$5.7e-05$	$2.0e-04$	$9.2e-04$	$1.4e-03$
0.09	—	$4.3e-09$	$1.1e-06$	$6.5e-06$	$2.6e-06$	$1.3e-04$	$2.0e-04$
0.07	—	$2.4e-11$	$2.5e-08$	$2.2e-07$	$1.1e-06$	$6.1e-06$	$9.3e-06$
0.05	—	$1.7e-15$	$3.0e-12$	$4.9e-10$	$3.5e-09$	$2.2e-08$	$3.5e-08$

Table 3: Using the subsolution based on quasipotential throughout. Estimated values for different pairs (ε, T) , one sided problem.

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	7	10
0.20	132	10	3	2	2	5	15
0.16	331	15	3	2	2	4	14
0.13	1418	23	4	2	2	4	14
0.11	3162	36	4	2	2	4	14
0.09	—	70	6	3	2	4	13
0.07	—	194	7	3	2	4	12
0.05	—	1300	12	4	2	4	12

Table 4: Using the subsolution based on quasipotential throughout. Relative errors per sample for different pairs (ε, T) , one sided problem.

which lead to poor performance of non-dynamic forms of importance sampling as discussed in [6, 11, 12].

It will turn out that to overcome the difficulties introduced by the rest point one must do a much better job of approximating the optimal change of measure than is possible based just on a time and ε -independent subsolution. However, it also turns out that the additional accuracy is needed only near the rest point, where in fact explicit time and ε -dependent solutions can be found. These are then combined with the simple time-independent subsolution based on the quasipotential to produce schemes that are nearly optimal and which protect against both sources of significant variance. An overview of the construction of such schemes is the topic of the next section.

4 Combining subsolutions with a refined local analysis: the linear problem

In this section we combine a local analysis that produces a time and ε -dependent scheme near the rest point with a scheme based on the quasipotential elsewhere. There are of course few process models and problems for which the related HJB equation can be solved explicitly. However, a class of processes where this is possible are the Gauss-Markov models, i.e., SDEs with drift that is linear in the state and constant diffusion matrix. For these processes and for terminal

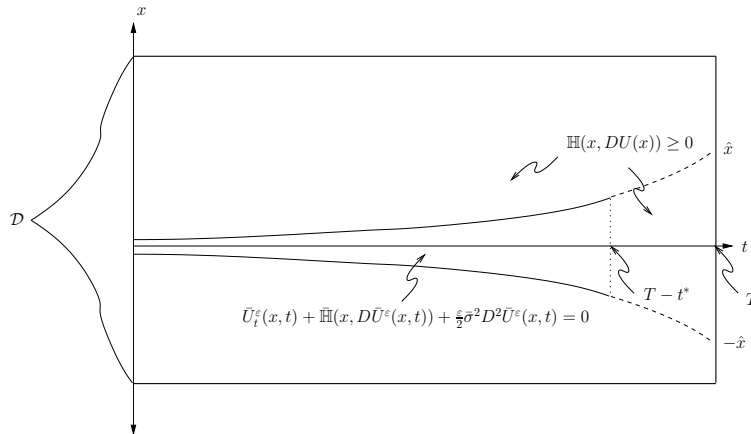


Figure 5: Partition of the state space in a combined scheme

conditions of the appropriate form, both the limit ($\varepsilon = 0$) PDE and the prelimit ($\varepsilon > 0$) PDE have an explicit solution that can be expressed in terms of the value function of a linear-quadratic regulator (LQR) control problem.

Our ultimate approach to the construction of importance sampling schemes is suggested by Figure 5. The problem of interest is of the form (2.1) or an analogous problem involving escape from \mathcal{D} prior to T . The particular problem will fix the boundary and terminal conditions on $[0, T] \times \partial\mathcal{D}$ and $\{T\} \times \mathcal{D}$. In the figure, we are interesting in estimating the exit probability

$$\theta^\varepsilon = \mathbb{P}_{0,0} \{X^\varepsilon \text{ hits } -A_1 \text{ or } A_2 \text{ before time } T\},$$

where 0 is the rest point and $\mathcal{D} = (-A_1, A_2)$ with $A_1, A_2 > 0$.

In most of the domain, which is the section outside the curves that terminate at $\pm\hat{x}$ and after time $T - t^*$, the control is based on a subsolution U constructed in terms of the quasipotential. Within the curves the solution to (1.1) is well approximated by a Gauss-Markov process. Hence within this region we will use a control that would be appropriate for a problem if the process were instead the approximating Gauss-Markov model. The function \bar{H} that defines the PDE for this region is therefore the one corresponding to the Gauss-Markov process. Besides the dynamics in this region (which are determined by the Gauss-Markov approximation), we must choose a terminal condition. This will be given by the minimum of two quadratic functions, one centered at \hat{x} and one centered at $-\hat{x}$. The parameter \hat{x} is a nominal value that for purposes of the present discussion can be taken to be 1. The parameter t^* plays two roles. One is to determine the size of the region on which the true dynamics are approximated by Gauss-Markov dynamics. The second role is related to the fact that if the control based on the quadratics centered at $\pm\hat{x}$ is used all the way to T then singularities will develop. For this reason, we switch back to the control based on the quasipotential after $T - t^*$, and t^* must be chosen so that the subsolution property is preserved across the handoff time $T - t^*$.

While the discussion above suggests the correct decomposition of the domain, the actual construction is more complex, since the control should transition nicely when moving between

the regions, and the construction of a scheme for which rigorous bounds can be proved will require additional mollification and approximations. However, the building blocks are always subsolutions to the indicated PDEs. The form of the surrogate problem for the Gauss-Markov model needs to be explained, as well as various boundary and terminal conditions. To simplify the discussion, we consider a sequence of successively more general problems. In this paper we complete the analysis for the one-dimensional problem. The multi-dimensional problem will be addressed in a companion paper.

4.1 A one-dimensional Gauss-Markov model

Our first goal is to construct a subsolution for the $\varepsilon = 0$ problem for the processes that will be used in the localization. This can be related to the problem of estimating the probability that the solution to

$$dX^\varepsilon(s) = -cX^\varepsilon(s)ds + \sqrt{\varepsilon}\bar{\sigma}dB(s), \quad X^\varepsilon(t) = x \in (-A, A)$$

escapes from the interval $[-A, A]$ before time T , a problem that was also used for the computational examples of the last subsection. The parameters c and $\bar{\sigma}$ are positive constants. In this subsection we will take $\hat{x} = A$, and because of this can postpone the issue regarding singularities in the control at $t = T$. We thus also take t^* to be zero, and will return to the role of t^* and its selection for a general problem in the next subsection. The corresponding PDE for the escape probability is

$$U_t^\varepsilon + \mathbb{H}(x, DU^\varepsilon) + \frac{\varepsilon}{2}\bar{\sigma}^2 D^2 U^\varepsilon = 0, \quad \mathbb{H}(x, p) = -cxp - \frac{1}{2}\bar{\sigma}^2 p^2,$$

plus the terminal and boundary conditions

$$U^\varepsilon(t, x) = \begin{cases} 0 & x = \pm A, t \in [0, T], \\ \infty & x \in (-A, A), t = T. \end{cases}$$

While simple in appearance, this equation does not have an explicit solution.

The equation obtained in the limit $\varepsilon \rightarrow 0$ is more tractable, and the unique viscosity solution can be described as follows. $U^0(t, x)$ corresponds to the variational problem

$$\inf \left\{ \int_t^T \frac{1}{2\bar{\sigma}^2} \left| \dot{\phi}(s) + c\phi(s) \right|^2 ds : \phi(t) = x, |\phi(s)| \geq A \text{ some } s \in [t, T] \right\}.$$

Depending on how and when the minimizing trajectory leaves $[-A, A]$, the solution takes a particular explicit form. (In all cases the minimizer can be found by solving the appropriate Euler-Lagrange equation.) If for the initial condition (t, x) the minimizing trajectory leaves before time T , then

$$U^0(t, x) = F_1(x) \doteq \frac{c}{\bar{\sigma}^2} [A^2 - x^2].$$

This is the case when the minimal cost is the negative of the quasipotential, translated by a constant to satisfy the boundary condition at the exit location. Such initial conditions satisfy

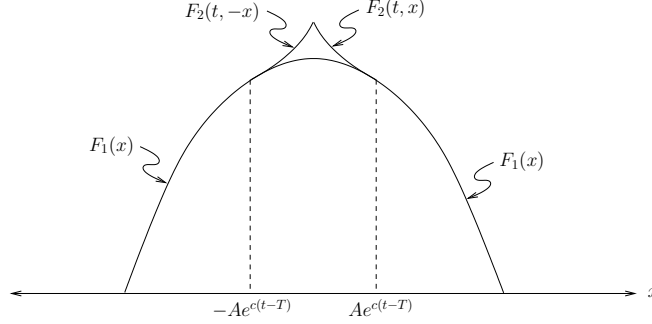


Figure 6: The $\varepsilon = 0$ solution for a fixed t .

$|x| \geq Ae^{c(t-T)}$. When $0 < x < Ae^{c(t-T)}$ the minimizer leaves through A at exactly time T , and the minimizing value is

$$U^0(t, x) = F_2(t, x) \doteq \frac{c}{\bar{\sigma}^2} \frac{(A - xe^{c(t-T)})^2}{[1 - e^{2c(t-T)}]}.$$

One can also interpret F_2 as the minimal cost for a linear quadratic regulator with a singular terminal cost applied at time T , i.e., a cost that equals 0 at A and ∞ otherwise.

By symmetry it is clear that when $x < 0$ the minimizing trajectory will exit at $-A$. Define

$$U_+^0(t, x) = \begin{cases} F_1(x) & , \text{ if } x \geq Ae^{c(t-T)} \\ F_2(t, x) & , \text{ if } 0 < x < Ae^{c(t-T)} \end{cases} \quad (4.1)$$

Setting $U_-^0(t, x) = U_+^0(t, -x)$, we have

$$U^0(t, x) = \begin{cases} U_+^0(t, x) & , \text{ if } x \geq 0 \\ U_-^0(t, x) & , \text{ if } x \leq 0 \end{cases}.$$

Note that when $x = Ae^{c(t-T)}$

$$F_2(t, Ae^{c(t-T)}) = \frac{c}{\bar{\sigma}^2} \frac{(A - Ae^{c(t-T)})^2}{[1 - e^{2c(t-T)}]} = \frac{c}{\bar{\sigma}^2} \frac{(A - Ae^{2c(t-T)})^2}{[1 - e^{2c(t-T)}]} = \frac{c}{\bar{\sigma}^2} [A^2 - x^2] = F_1(Ae^{c(t-T)}),$$

and therefore $U^0(t, x)$ is continuous for all $(t, x) \in [0, T) \times [-A, A]$. In fact more is true, and one can check that

$$DF_2(t, Ae^{c(t-T)}) = DF_1(Ae^{c(t-T)}),$$

and thus $U^0(t, x)$ has a continuous partial derivative in x for $x \in (0, A)$. The mapping $x \rightarrow F_2(t, x)$ is convex, and the graph of this mapping lies above that of $F_1(x)$, which is concave. Thus the two graphs intersect only at the point $(Ae^{c(t-T)}, cA^2[1 - e^{2c(t-T)}]/\bar{\sigma}^2)$, where both functions and their first derivatives in x agree. See Figure 6. Note also that $\tilde{U}^0(t, x) \doteq U_+^0(t, x)$ for $x \in (-\infty, A]$ is the solution to the $\varepsilon = 0$ problem with escape from $(-\infty, A]$ (a “one-sided” version of the problem of escape from $[-A, A]$).

Simulation data for the schemes based on these two value functions are presented below. The approximated values are omitted since they are qualitatively similar to those in Tables 1 and 3, and only relative errors are presented. Table 5 gives data based on U^0 for the problem of two-sided exit, and should be compared to Table 2. The use of the solution to HJB equation for $\varepsilon = 0$ drastically improves the performance for small T , which is due to the fact that the subsolution based on the quasipotential is a very poor approximation to the solution for $\varepsilon > 0$ for such T . However, for large T the two schemes are comparably bad.

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	10	14	18
0.20	2	2	1	1	1	10	57	266
0.16	3	3	1	1	1	10	55	265
0.13	4	2	1	1	1	9	51	394
0.11	3	2	2	1	1	9	44	177
0.09	3	3	2	1	1	9	43	314
0.07	3	3	2	2	1	9	67	520
0.05	4	2	2	2	1	8	50	278

Table 5: Using the subsolution based on the explicit solution to the $\varepsilon = 0$ HJB equation. Relative errors per sample for different pairs (ε, T) , two sided problem.

The data for exit from one side are given in Table 6. In contrast with the two-sided problem, the performance does not degrade so quickly with large T . This suggests that the decline in performance is not so much due to using the approximating $\varepsilon = 0$ PDE, but rather the possible lack of regularity of the solution to this equation. Recall that the difficulty with the use of the

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	7	10
0.20	1	1	1	1	1	2	3
0.16	1	1	1	1	1	2	4
0.13	1	1	1	1	1	2	3
0.11	1	1	1	1	1	2	3
0.09	1	1	1	1	1	2	3
0.07	1	1	1	1	1	2	3
0.05	1	1	1	1	1	2	3

Table 6: Using the subsolution based on the explicit solution to the $\varepsilon = 0$ HJB equation. Relative errors per sample for different pairs (ε, T) , one sided problem.

subsolution based on the quasipotential alone (here $F_1(x)$) was that in a neighborhood of $x = 0$ the second derivative was negative, and when T is large this leads to poor control of the variance of the associated scheme for both the one-sided and two-sided problems.

When using the time-dependent $\varepsilon = 0$ PDE as the basis for a scheme for the one-sided problem, the introduction of $F_2(t, x)$ appears to have largely mitigated the problem due to the second derivative term. Note that this function determines the value of $\hat{U}^0(t, x)$ when $x = 0$ and is convex rather than concave in x . In contrast, for the two-sided problem there is a concave singularity at the origin. There the second derivative in x is $-\infty$, and the subsolution property

for the $\varepsilon > 0$ problem again fails. What is needed is a subsolution that works across the point $x = 0$ for the $\varepsilon > 0$ problem. Such a subsolution can be constructed using the mollification introduced in the next subsection. Simulations based on such a mollified version of the $\varepsilon = 0$ subsolution (and with mollification parameter $\delta = 2\varepsilon$) are presented in Table 7, and support the claim just made.

$\varepsilon \mid T$	0.25	0.5	1	1.5	2.5	10	14	18	23
0.20	1	1	1	1	1	1	3	5	61
0.16	1	1	1	1	1	1	3	5	68
0.13	1	1	1	1	1	1	2	4	64
0.11	2	1	1	1	1	1	2	4	63
0.09	2	2	2	2	1	1	2	4	58
0.07	2	2	2	2	1	1	2	3	55
0.05	2	2	2	2	1	2	2	3	52

Table 7: Using the subsolution based on the mollification of $U_+^0(t, x) \wedge U_-^0(t, x)$. Relative errors per sample with $\hat{x} = A = 1$, two sided problem.

In spite of the shortcomings we have described in this section, the solution to the $\varepsilon = 0$ problem serves as the starting point for a construction that can be shown to perform well both in theory and in practice. There are three important modifications that are needed.

- The first is that, in order to effectively deal with the $\varepsilon > 0$ dynamics and in particular to avoid the degradation still present in Table 6 when T is large, the region where the F_2 subsolution determines the dynamics must be enlarged. The solution to the $\varepsilon = 0$ problem constructed above leads to a region that vanishes exponentially in $t - T$ [see (4.1)].
- The second modification will be the use a mollification to eliminate singularities such as the one at $x = 0$ and help guarantee a global subsolution property for the $\varepsilon > 0$ PDE. The particular mollification we use is very convenient, and was first used for importance sampling in [7], though with a somewhat different intended use.
- The final modification was also alluded to previously, which is to revert to the quasipotential based control in an interval of the form $[T - t^*, T]$. All three modifications will be introduced in the next subsection in the context of the Gauss-Markov process.

4.2 Simulation scheme for the Gauss-Markov model

In this subsection we generalize the construction of the last subsection. As discussed there, the generalizations are needed to address issues that play a role in both the theoretical analysis of the scheme and its practical performance. After introducing these generalizations, we will demonstrate (numerically and theoretically) that the suggested change of measure does not degrade in performance as T gets larger and is close to optimality not only as $\varepsilon \downarrow 0$, but for fixed $\varepsilon > 0$ as well.

We begin by introducing the first generalization, which is to replace the terminal condition $\infty \cdot (x - \hat{x})^2 + F_1(\hat{x})$, which was used to define $F_2(t, x)$, by a terminal condition of the form

$M(x - \hat{x})^2/2 + F_1(\hat{x})$. The parameter \hat{x} replaces A and is a nominal value introduced to disconnect the localization from the boundary. The solution to the LQR that corresponds to $M(x - \hat{x})^2/2 + F_1(\hat{x})$, which will be denoted $F_2^M(t, x)$, is automatically smaller than $F_2(t, x) = F_2^\infty(t, x)$. The motivation for replacing ∞ by $M/2$ is because we want the solution to the LQR problem [i.e., $F_2^M(t, x)$] to determine the control near $x = 0$ for a set whose width is uniformly (in t) bounded below away from zero. As discussed in the last section, the second derivative term associated with F_1 is of the wrong sign and degrades performance. The neighborhood where $F_2^M(t, x) < F_1(x)$ does not degenerate as $T - t \rightarrow \infty$, and its size is decreasing in M . The introduction of M complicates the construction by also requiring mollification (in addition to the one that will be needed at $x = 0$), since F_2^M can no longer be smoothly merged with F_1 .

For $M \in (0, \infty)$ the solution to this LQR takes the form

$$F_2^M(t, x) = a^M(t) \left(x - \hat{x} e^{-c(t-T)} \right)^2 + F_1(\hat{x}), \quad a^M(t) = \frac{c e^{2c(t-T)}}{(2c/M + \bar{\sigma}^2) - \bar{\sigma}^2 e^{2c(t-T)}} > 0.$$

Recall that $F_1(x) = c[A^2 - x^2]/\bar{\sigma}^2$ so that $F_1(A) = 0$. Define $F_{2,+}^M(t, x) = F_2^M(t, x)$, $F_{2,-}^M(t, x) = F_2^M(t, -x)$. It will be important to know which of $F_{2,+}^M$, $F_{2,-}^M$, and F_1 is smallest, and we note here several properties. Let $K \doteq 2c/M + \bar{\sigma}^2$. The first is that there are two real solutions to $F_{2,+}^M(t, x) = F_1(x)$, and these take the form

$$\frac{\bar{\sigma}^2 \hat{x}}{K} \left(e^{c(t-T)} \pm \sqrt{\frac{2cK}{M\bar{\sigma}^4} - \frac{2c}{M\bar{\sigma}^2} e^{2c(t-T)}} \right). \quad (4.2)$$

Between these roots $F_{2,+}^M(t, x) < F_1(x)$, and on the complement of the interval the reverse inequality holds. The limit $t - T \rightarrow -\infty$ gives the asymptotic endpoints of the interval where $F_{2,+}^M(t, x) < F_1(x)$, which are

$$\pm \hat{x} \sqrt{\frac{2c}{2c + M\bar{\sigma}^2}}.$$

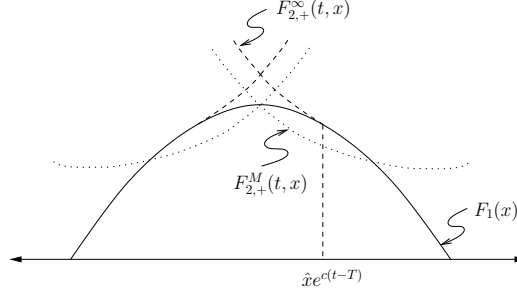
We point out here that a natural scaling for this problem, given that the size of the neighborhood of zero where the quasipotential based subsolution fails for the $\varepsilon > 0$ system scales like $\sqrt{\varepsilon}$, is to ask that this width scale as $2\varepsilon^\kappa$, $\kappa \in [0, 1/2]$. If for example the desired width is $2\varepsilon^{1/4}$, then when $\varepsilon > 0$ is small then we should take $M \approx 2\hat{x}^2 c / \bar{\sigma}^2 \varepsilon^{1/2}$.

The next adaptation is required so that singularities in the control associated with $F_{2,\pm}^M(t, x)$ as $t \uparrow T$ do not cause a problem, as well as for purposes of localization. For a parameter $t^* \geq 0$ we will want $U^0(T - t^*, x) \leq F_1(x)$ for all $x \in [-A, A]$, where (with an abuse of notation) $U^0(t, x) \doteq F_{2,+}^M(t, x) \wedge U_+^0(t, x) \wedge F_{2,-}^M(t, x) \wedge U_-^0(t, x)$. This is true if and only if the smaller solution to $F_{2,+}^M(T - t^*, x) = F_1(x)$ is less than or equal to zero. This root was found to be

$$\frac{\bar{\sigma}^2 \hat{x}}{K} \left(e^{-ct^*} - \sqrt{\frac{2cK}{M\bar{\sigma}^4} - \frac{2c}{M\bar{\sigma}^2} e^{-2ct^*}} \right),$$

and the restriction that this be non positive can be simplified to

$$t^* \geq -\frac{1}{2c} \log \frac{2c}{M\bar{\sigma}^2}.$$

Figure 7: Relations between $F_1, F_{2,\pm}^\infty$ and $F_{2,\pm}^M$.

The inequality $U^0(T - t^*, x) \leq F_1(x)$ will ensure that the subsolution property is preserved if we switch from using U^0 for $t \leq T - t^*$ to using F_1 for $t > T - t^*$. If $T \leq t^*$ this means that we always use F_1 , but our interest here is in large T .

We assume that $M \geq 4c/\bar{\sigma}^2$ so that $t^* > 0$. To guarantee that the smaller root is strictly negative and to conveniently satisfy a bound used later on we assume

$$t^* \geq -\frac{2}{c} \log \frac{2c}{M\bar{\sigma}^2}. \quad (4.3)$$

Besides enforcing the subsolution property across the handoff at time $T - t^*$, the selection of t^* plays a key role in determining the region used for the localization for the general nonlinear problem. Owing to the exponential decay, one can confine the localization to a small region with a modest value of t^* . Suppose we consider confining to a region that scales as $2\varepsilon^{1/4}$ for all $t \in [0, T - t^*]$ and arbitrarily large T . As discussed previously, for small t and large $T - t^*$ this suggests that M be approximately $2\hat{x}^2 c / \bar{\sigma}^2 \varepsilon^{1/2}$, which means that $t^* \geq -\frac{2}{c} \log[\varepsilon^{1/2} / \hat{x}^2]$. Recalling that the loss in performance of the importance sampling scheme when the quasipotential based subsolution is used scales like εc , this gives a loss over such an interval of the form $[T - t^*, T]$ as scaling like $-\varepsilon \log[\varepsilon / \hat{x}^2]$. We will return to such considerations in the next section.

Finally there is the issue of mollification. Owing to the replacement of $F_{2,\pm}(t, x)$ by $F_{2,\pm}^M(t, x)$, there are now several sources of discontinuity in the gradient. The subsolution prior to mollification would in general take the form $F_{2,+}^M(t, x) \wedge U_+^0(t, x) \wedge F_{2,-}^M(t, x) \wedge U_-^0(t, x)$. However, since we know that $F_{2,+}^M(t, x) \wedge F_{2,-}^M(t, x)$ is dominated by $F_1(x)$ near $x = 0$, we can also use

$$F_{2,+}^M(t, x) \wedge F_{2,-}^M(t, x) \wedge F_1(x).$$

See Figure 7.

We next state a result that will be used to derive performance bounds for schemes based on the mollification. The proof is deferred to the Appendix. We consider the general one dimensional process model

$$dX^\varepsilon = b(X^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X^\varepsilon)dB,$$

where b and σ are Lipschitz continuous. Letting $\alpha(x) = \sigma(x)^2$, the relevant ε -dependent PDE is

$$\mathcal{G}^\varepsilon[U](t, x) = U_t(t, x) + DU(t, x)b(x) - \frac{1}{2}|\sigma(x)DU(t, x)|^2 + \frac{\varepsilon}{2}\alpha(x)D^2U(t, x) = 0.$$

Lemma 4.1 *Suppose that the functions $\tilde{U}_i : [0, T] \times \mathbb{R}$, $i = 1, \dots, n$ are twice continuously differentiable in x , once continuously differentiable in t , and satisfy*

$$\mathcal{G}^\varepsilon[\tilde{U}_i](t, x) \geq \gamma_i(x, t, \varepsilon).$$

For $\delta > 0$ define

$$U^\delta(t, x) = -\delta \log \left(\sum_{i=1}^n e^{-\frac{1}{\delta} \tilde{U}_i(t, x)} \right),$$

and define the weights

$$\rho_i(t, x; \delta) = \frac{e^{-\frac{1}{\delta} \tilde{U}_i(t, x)}}{\sum_{i=1}^n e^{-\frac{1}{\delta} \tilde{U}_i(t, x)}}.$$

Then

$$\min \left\{ \tilde{U}_i(t, x), i = 1, \dots, n \right\} \geq U^\delta(t, x) \geq \min \left\{ \tilde{U}_i(t, x), i = 1, \dots, n \right\} - \delta \log n,$$

and for $0 < \varepsilon \leq \delta$

$$\begin{aligned} \mathcal{G}^\varepsilon[U^\delta](t, x) &\geq \frac{1}{2} \left(1 - \frac{\varepsilon}{\delta} \right) \left[\sum_{i=1}^n \rho_i(t, x; \delta) \left| \sigma(x) D \tilde{U}_i(t, x) \right|^2 - \sum_{i=1}^n \rho_i(t, x; \delta) \left| \sigma(x) D \tilde{U}_i(t, x) \right|^2 \right] \\ &\quad + \sum_{i=1}^n \rho_i(t, x; \delta) \gamma_i(x, t, \varepsilon) \\ &\geq \sum_{i=1}^n \rho_i(t, x; \delta) \gamma_i(x, t, \varepsilon). \end{aligned}$$

Based on this result we consider the mollification of $U^0(t, x) \doteq F_{2,+}^M(t, x) \wedge F_{2,-}^M(t, x) \wedge F_1(x)$, which is

$$U^\delta(t, x) = -\delta \log \left(e^{-\frac{1}{\delta} F_{2,+}^M(t, x)} + e^{-\frac{1}{\delta} F_{2,-}^M(t, x)} + e^{-\frac{1}{\delta} F_1(x)} \right).$$

For reasons that will be made clear in the analysis (see Lemma 4.5), we generally use $\delta = 2\varepsilon$.

As discussed previously, we return to the control based on the quasipotential for the last t^* units of time, and so the subsolution takes the form

$$\bar{U}^\delta(t, x) = \begin{cases} F_1(x), & t > T - t^* \\ U^\delta(t, x), & t \leq T - t^* \end{cases} \quad (4.4)$$

Note that the mollification reduces values, in that $U^\delta(t, x) \leq U^0(t, x)$, and so the requirement $U^\delta(T - t^*, x) \leq F_1(x)$ for $x \in [-A, A]$ holds, since $U^0(T - t^*, x) \leq F_1(x)$.

4.3 Analysis of the scheme

We next present a rigorous and nonasymptotic bound for the second moment of the importance sampling scheme constructed in the last subsection. To derive a bound that is valid for $\varepsilon > 0$ and uniform in T , we use the same representation as in (3.2). In particular, we choose $\bar{U}(t, x) = \bar{U}^\delta(t, x)$ defined via (4.4) for the design and $\bar{W}(t, x) = \bar{U}^{\delta, \eta}(t, x)$ for the analysis, where

$$\bar{U}^{\delta, \eta}(t, x) = \begin{cases} F_1(x), & t > T - t^* \\ U^{\delta, \eta}(t, x), & t \leq T - t^* \end{cases},$$

with

$$U^{\delta, \eta}(t, x) = (1 - \eta)U^\delta(t, x).$$

As discussed in Section 3 this choice is driven by the need for a subsolution for the $\varepsilon > 0$ dynamics with an explicit form, and this limitation of the technique leads to conservative bounds on the true performance. To simplify notation, for smooth functions W, U we define

$$\mathcal{G}^\varepsilon[W, U](t, x) = \mathcal{G}^\varepsilon[W](t, x) - \frac{1}{2} |\bar{\sigma} (DW(t, x) - DU(t, x))|^2, \quad (4.5)$$

where $\mathcal{G}^\varepsilon[W]$ is defined in (4.2). Using that $\mathcal{G}^0[F_1] = \mathcal{G}^0[F_2^M] = 0$,

$$\mathcal{G}^\varepsilon[F_1](x) = \frac{\varepsilon}{2} \bar{\sigma}^2 D^2 F_1(x) = -\varepsilon c \text{ and } \mathcal{G}^\varepsilon[F_2^M](t, x) = \frac{\varepsilon}{2} \bar{\sigma}^2 D^2 F_2^M(t, x) = \varepsilon \bar{\sigma}^2 a^M(t).$$

Since the problem is symmetric, it suffices to consider only $x \in [0, A]$. A key role is played by the weights associated with the exponential mollification of the subsolution, which take the forms

$$\rho_2^{M, \pm}(t, x; \delta) = \frac{e^{-\frac{1}{\delta} F_{2, \pm}^M(t, x)}}{e^{-\frac{1}{\delta} F_{2, +}^M(t, x)} + e^{-\frac{1}{\delta} F_{2, -}^M(t, x)} + e^{-\frac{1}{\delta} F_1(x)}}$$

and

$$\rho_1(t, x; \delta) = \frac{e^{-\frac{1}{\delta} F_1(x)}}{e^{-\frac{1}{\delta} F_{2, +}^M(t, x)} + e^{-\frac{1}{\delta} F_{2, -}^M(t, x)} + e^{-\frac{1}{\delta} F_1(x)}}.$$

To determine which of these dominate at any (t, x) , the relative sizes of the functions $F_{2, \pm}^M(t, x)$ and $F_1(x)$ are required, with smaller functions corresponding to more dominant weights. For this reason the solutions to $F_{2, \pm}^M(t, x) = F_1(x)$ play an important role, and especially the larger one identified in (4.2). See Figure 7. The following bounds on this root will be used to partition the domain in the analysis. Let $z \doteq \hat{x}(c/M\bar{\sigma}^2)^{1/2}/2$ and let $R(t)$ denote the larger root in (4.2). Then we claim under (4.3) that

$$2z \leq R(t) \leq 8z \text{ for all } t \in [0, T - t^*]$$

and uniformly in T . We recall the definition $K \doteq (2c/M) + \bar{\sigma}^2 \in (\bar{\sigma}^2, \infty)$ and that $M \geq 4c/\bar{\sigma}^2$, so that $K \in (\bar{\sigma}^2, 2\bar{\sigma}^2)$. Then the smallest possible value of $R(t)$ satisfies

$$\frac{\bar{\sigma}^2 \hat{x}}{K} \sqrt{\frac{2cK}{M\bar{\sigma}^4}} > \frac{\bar{\sigma}^2 \hat{x}}{\sqrt{2}\bar{\sigma}} \sqrt{\frac{2c}{M\bar{\sigma}^4}} = \hat{x} \sqrt{\frac{c}{M\bar{\sigma}^2}} = 2z,$$

while the largest satisfies

$$\begin{aligned}
\frac{\bar{\sigma}^2 \hat{x}}{K} \left(e^{-ct^*} + \sqrt{\frac{2cK}{M\bar{\sigma}^4} - \frac{2c}{M\bar{\sigma}^2} e^{-2ct^*}} \right) &\leq \frac{\bar{\sigma}^2 \hat{x}}{K} \left(\frac{2c}{M\bar{\sigma}^2} + \sqrt{\frac{2cK}{M\bar{\sigma}^4}} \right) \\
&\leq \frac{\bar{\sigma}^2 \hat{x}}{\bar{\sigma}^2} \left(\frac{2c}{M\bar{\sigma}^2} + \sqrt{\frac{4c}{M\bar{\sigma}^2}} \right) \\
&< \hat{x} \left(4\sqrt{\frac{c}{M\bar{\sigma}^2}} \right) \\
&= 8z,
\end{aligned}$$

where the first inequality uses $t^* \geq -\frac{2}{c} \log \frac{2c}{M\bar{\sigma}^2}$. We set $H \doteq 10z > 8z$.

In order to obtain bounds on the performance under the corresponding scheme, we need to bound $\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x)$ from below in various regions. By (4.5)

$$\begin{aligned}
\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) &= \mathcal{G}^\varepsilon[U^{\delta,\eta}](t, x) - \frac{1}{2} \left| \bar{\sigma} \left(DU^{\delta,\eta}(t, x) - DU^\delta(t, x) \right) \right|^2 \\
&= \mathcal{G}^\varepsilon[U^{\delta,\eta}](t, x) - \frac{1}{2} \eta^2 \left| \bar{\sigma} DU^\delta(t, x) \right|^2.
\end{aligned} \tag{4.6}$$

We will use the notation

$$\gamma_1 = \mathcal{G}^\varepsilon[F_1](x) = -\varepsilon c$$

and

$$\gamma_2^M(t) = \mathcal{G}^\varepsilon[F_2^M](t, x) = \varepsilon \bar{\sigma}^2 a^M(t).$$

Straightforward calculations and some algebra give

$$\mathcal{G}^\varepsilon[U^{\delta,\eta}](t, x) \geq (1 - \eta) \mathcal{G}^\varepsilon[U^\delta](t, x) + \frac{1}{2} (\eta - \eta^2) \left| \bar{\sigma} DU^\delta(t, x) \right|^2.$$

For notational convenience, define

$$\begin{aligned}
\beta_0(t, x) &= \bar{\sigma}^2 \left[\rho_2^{M,+} |DF_{2,+}^M|^2 + \rho_2^{M,-} |DF_{2,-}^M|^2 + \rho_1 |DF_1|^2 \right. \\
&\quad \left. - \left| \rho_2^{M,+} DF_{2,+}^M + \rho_2^{M,-} DF_{2,-}^M + \rho_1 DF_1 \right|^2 \right] (t, x).
\end{aligned}$$

Note that by Jensen's inequality $\beta_0(t, x) \geq 0$. We next apply Lemma 4.1 to $\mathcal{G}^\varepsilon[U^{\delta,\eta}](t, x)$ (while suppressing the dependence on δ in the notation for the ρ 's), and use (4.6) to get

$$\begin{aligned}
\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) &\geq (1 - \eta) \frac{1}{2} \left(1 - \frac{\varepsilon}{\delta} \right) \beta_0(t, x) + (1 - \eta) \left[\rho_2^{M,+}(t, x) + \rho_2^{M,-}(t, x) \right] \gamma_2^M(t) \\
&\quad + (1 - \eta) \rho_1(t, x) \gamma_1 + \frac{1}{2} (\eta - 2\eta^2) \left| \bar{\sigma} DU^\delta(t, x) \right|^2
\end{aligned} \tag{4.7}$$

for all $x \in [-A, A]$ and $t \in [0, T - t^*]$.

We will partition the domain according to $z \doteq \hat{x}(c/M\bar{\sigma}^2)^{1/2}/2$ and $H \doteq 10z$. We consider three cases depending on whether $x \in [0, z]$, $x \in [z, H]$ or $x \in [H, A]$ if $x \geq 0$. The case $x < 0$ is symmetric. Before proceeding with the analysis for each of the cases, we give the definition of exponential negligibility, a concept used frequently in the rest of the paper.

Definition 4.2 A term is called *exponential negligible* if it is bounded above in absolute value by a quantity of the form $\varepsilon c e^{-\frac{d}{\varepsilon}}$, where $c < \infty$ and $d > 0$.

Lemma 4.3 Assume that $(t, x) \in [0, T - t^*] \times [0, z]$, $\delta \geq \varepsilon$ and $\eta \leq 1/2$. Then, up to an exponentially negligible term

$$\mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq 0.$$

Proof. In this region $F_1(x) \geq F_{2,+}^M(t, x)$, and we claim that the inequality is in fact strict. We have

$$F_1(x) - F_{2,+}^M(t, x) = \frac{c}{\bar{\sigma}^2} [\hat{x}^2 - x^2] - \frac{c}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left[x e^{c(t-T)} - \hat{x} \right]^2. \quad (4.8)$$

For each fixed t this defines a concave function of x . At $x = 0$ the value is minimized when $t = T - t^*$. Using $e^{-2ct^*} \leq [2c/M\bar{\sigma}^2]^4 \leq c/M\bar{\sigma}^2$ (since $M \geq 4c/\bar{\sigma}^2$) and $K = \bar{\sigma}^2 + 2c/M$, we obtain the strictly positive lower bound $\hat{x}^2 c [1/\bar{\sigma}^2 - 1/[\bar{\sigma}^2 + c/M]]$. Since $F_1(2z) - F_{2,+}^M(t, 2z) \geq 0$, by concavity there is $c_1 > 0$ such that (4.8) is bounded below by c_1 for all $(t, x) \in [0, T - t^*] \times [0, z]$. Thus the term in (4.7) involving the weight ρ_1 is exponentially negligible. Since $\beta_0(t, x) \geq 0$, $\gamma_2^M(t) > 0$ and $\eta \leq 1/2$, all other terms are non-negative, and the result follows. ■

Lemma 4.4 Assume that $(t, x) \in [0, T - t^*] \times [H, A]$, $\delta \geq \varepsilon$ and $\eta \leq 1/4$. Then letting $\varepsilon_0 = cH^2/3\bar{\sigma}^2$, we have that for all $\varepsilon \in (0, \varepsilon_0)$ and with any $\eta \in [\varepsilon/(\varepsilon + cH^2/\bar{\sigma}^2), 1/4]$, up to an exponentially negligible term

$$\mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq 0. \quad (4.9)$$

Proof. In this region $F_{2,+}^M(t, x) \geq F_1(x)$, and it is straightforward that the terms associated with $F_{2,-}^M(t, x)$ are exponentially negligible. Note that $x \rightarrow F_{2,+}^M(t, x) - F_1(x)$ is convex, recall that for each $t \in [0, T - t^*]$ the largest value where the two functions agree is smaller than $8z \leq H$, and that

$$DF_{2,+}^M(t, x) - DF_1(x) = \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(x - \frac{\bar{\sigma}^2}{K} \hat{x} e^{c(t-T)} \right). \quad (4.10)$$

Inserting the largest root for the given t gives the value

$$\frac{2c\hat{x}}{K - \bar{\sigma}^2 e^{2c(t-T)}} \sqrt{\frac{2cK}{M\bar{\sigma}^4} - \frac{2c}{M\bar{\sigma}^2}} e^{-2ct^*}.$$

A lower bound on the first term is $2c\hat{x}/K$. Using $e^{-ct^*} \leq [2c/M\bar{\sigma}^2]^2$, the definition of K , and $4c/M\bar{\sigma}^2 \leq 1$ to bound the second term from below produces the strictly positive lower bound

$$DF_{2,+}^M(t, x) - DF_1(x) \geq \frac{2c\hat{x}}{K} \sqrt{\frac{2c}{M\bar{\sigma}^2} \left(\frac{c}{M} + \bar{\sigma}^2 \right)}$$

for all $t \in [0, T - t^*]$ and $x \geq 8z$. Since $H = 10z > 8z$, this shows that there is $c_2 > 0$ such that $F_{2,+}^M(t, x) - F_1(x) \geq c_2$ for all $(t, x) \in [0, T - t^*] \times [H, A]$. It follows that terms involving

$\rho_2^{M,\pm}(t, x)$ are exponentially negligible. Since $\beta_0(t, x) \geq 0$ and $\rho_1(t, x) = 1$ up to an exponentially negligible term,

$$\begin{aligned} \mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) &\geq (1 - \eta)\rho_1(t, x)\gamma^0 + \frac{1}{2}(\eta - 2\eta^2)\bar{\sigma}^2|\rho_1(t, x)DF_1(x)|^2 \\ &\geq -(1 - \eta)\varepsilon c + 2(\eta - 2\eta^2)\frac{c^2}{\bar{\sigma}^2}x^2 \end{aligned}$$

up to an exponentially negligible term. Choosing $\eta \leq 1/4$ gives

$$\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq -(1 - \eta)\varepsilon c + \eta\frac{c^2}{\bar{\sigma}^2}x^2$$

and for ε small enough such that $\eta \in [\varepsilon/(\varepsilon + cH^2/\bar{\sigma}^2), 1/4]$, the last display is non-negative. For this interval to be nonempty imposes the constraint $\varepsilon \leq \varepsilon_0 \doteq cH^2/3\bar{\sigma}^2$. Hence in this region and up to an exponentially negligible term,

$$\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq 0.$$

■

The final region is the most difficult, since $F_1(x) - F_{2,+}^M(t, x)$ can be either positive or negative.

Lemma 4.5 *Assume that $(t, x) \in [0, T - t^*] \times [z, H]$, $\eta \leq 1/4$ and set $\delta = 2\varepsilon$. Then up to an exponentially negligible term*

$$\mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq \frac{1}{2} \left[\frac{c^2\eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{c(t-T)} \right)^2 - 2\varepsilon c \right] \wedge 0.$$

Proof. While terms corresponding to $F_{2,-}^M(t, x)$ are exponentially negligible in this region, since

$$F_1(x) - F_{2,+}^M(t, x)$$

changes sign both $\rho_2^{M,+}$ and ρ_1 may be important. Since they are negligible we omit terms corresponding to $F_{2,-}^M(t, x)$.

By (4.7) we have up to an exponentially negligible term

$$\begin{aligned} \mathcal{G}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) &\geq (1 - \eta)\frac{1}{4}\beta_0(t, x) + (1 - \eta)\rho_2^{M,+}(t, x)\gamma_2^M(t) + (1 - \eta)\rho_1(t, x)\gamma_1 \\ &\quad + \frac{1}{2}(\eta - 2\eta^2)\bar{\sigma}^2 \left| \rho_2^{M,+}(t, x)DF_2^M(t, x) + \rho_1(t, x)DF_1(x) \right|^2 \end{aligned} \quad (4.11)$$

As noted previously $\beta_0(t, x) \geq 0$ for all (t, x) . However, we will exploit the fact that $\beta_0(t, x) = 0$ only for points (t, x) such that $DF_1(x) = DF_2^M(t, x)$. We distinguish two cases depending on whether $\rho_1(t, x) > 1/2$ or $\rho_1(t, x) \leq 1/2$.

Case I: $\rho_1(t, x) > 1/2$. We know that $\beta_0(t, x) \geq 0$ and $\gamma_2^M(t) \geq 0$ and can ignore those terms. Using $\rho_2^{M,+} + \rho_1 = 1$, the terms that remain are

$$\begin{aligned} (1 - \eta)\rho_1(t, x)\gamma_1 + \frac{1}{2}(\eta - 2\eta^2)\bar{\sigma}^2 \left[\rho_1(t, x)^2 |DF_1(x) - DF_{2,+}^M(t, x)|^2 \right. \\ \left. + 2\rho_1(t, x)DF_{2,+}^M(t, x)(DF_1(x) - DF_{2,+}^M(t, x)) + |DF_{2,+}^M(t, x)|^2 \right]. \end{aligned}$$

We claim that for $(t, x) \in [0, T - t^*] \times [z, H]$

$$DF_{2,+}^M(t, x)(DF_1(x) - DF_{2,+}^M(t, x)) \geq 0.$$

First, we note that $e^{-ct^*} \leq (2c/M\bar{\sigma}^2)^{1/2} = (4c/M\bar{\sigma}^2)^{1/2}/4 \leq 1/4$, and thus $\hat{x}e^{ct^*} \geq 4\hat{x} \geq H$. Therefore

$$\begin{aligned} DF_{2,+}^M(t, x) &= \frac{2ce^{2c(t-T)}}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(x - \hat{x}e^{-c(t-T)} \right) \\ &\leq \frac{2ce^{2c(t-T)}}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(H - \hat{x}e^{ct^*} \right) \\ &\leq 0. \end{aligned}$$

Second, by (4.10), the definition of z and $e^{-ct^*} \leq (2c/M\bar{\sigma}^2)^2$, we also have

$$\begin{aligned} DF_1(x) - DF_{2,+}^M(t, x) &= \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\frac{\bar{\sigma}^2}{K} \hat{x}e^{c(t-T)} - x \right) \\ &\leq \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\frac{\bar{\sigma}^2}{K} \hat{x}e^{-ct^*} - z \right) \\ &\leq \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\hat{x} \left(\frac{2c}{M\bar{\sigma}^2} \right)^2 - \frac{\hat{x}}{2} \left(\frac{c}{M\bar{\sigma}^2} \right)^{1/2} \right) \\ &\leq 0, \end{aligned}$$

where the last inequality uses $4c/M\bar{\sigma}^2 \leq 1$. We conclude that $DF_2^M(t, x)(DF_1(x) - DF_2^M(t, x)) \geq 0$. Since $\rho_1(t, x) \in (1/2, 1)$ and $\eta \leq 1/4$, we obtain the bound

$$\mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq -(1 - \eta)\varepsilon c + \frac{1}{16}\eta\bar{\sigma}^2 |DF_1(x) - DF_{2,+}^M(t, x)|^2. \quad (4.12)$$

This gives a bound for Case I.

Case II: $\rho_1(t, x) \leq 1/2$. Here we will have to use $\beta_0(t, x)$. Dropping other terms on the right that are not possibly negative, we obtain from (4.11) that

$$\mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq (1 - \eta)\frac{1}{4}\beta_0(t, x) + (1 - \eta)\rho_1(t, x)\gamma_1.$$

Omitting exponentially negligible terms, we note that

$$\begin{aligned} \beta_0(t, x) &= \bar{\sigma}^2 \left[(1 - \rho_1) |DF_{2,+}^M|^2 + \rho_1 |DF_1|^2 - |DF_{2,+}^M + \rho_1(DF_1 - DF_{2,+}^M)|^2 \right] (t, x) \\ &= \bar{\sigma}^2 \rho_1 \left[|DF_1|^2 - |DF_{2,+}^M|^2 - 2DF_{2,+}^M(DF_1 - DF_{2,+}^M) - \rho_1(DF_1 - DF_{2,+}^M)^2 \right] (t, x) \\ &= \bar{\sigma}^2 \rho_1(DF_1 - DF_{2,+}^M) [DF_1 + DF_{2,+}^M - DF_{2,+}^M - \rho_1(DF_1 - DF_{2,+}^M)] (t, x) \\ &= \bar{\sigma}^2 \rho_1(1 - \rho_1)(DF_1 - DF_{2,+}^M)^2(t, x) \\ &\geq \frac{1}{2}\bar{\sigma}^2 \rho_1(DF_1 - DF_{2,+}^M)^2(t, x), \end{aligned}$$

where the last inequality uses $\rho_1(t, x) \leq 1/2$, and so obtain

$$\mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq (1 - \eta)\rho_1 \left[\frac{1}{8}\bar{\sigma}^2 |DF_1(x) - DF_{2,+}^M(t, x)|^2 - \varepsilon c \right].$$

This gives a bound for Case II.

Straightforward estimation gives, for all $(t, x) \in [0, T - t^*] \times [z, H]$, the lower bound

$$\begin{aligned} DF_{2,+}^M(t, x) - DF_1(x) &= \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(x - \frac{\bar{\sigma}^2}{K} \hat{x} e^{c(t-T)} \right) \\ &\geq \frac{2c}{\bar{\sigma}^2} \left(z - \hat{x} e^{c(t-T)} \right). \end{aligned}$$

Using this bound and $\eta \leq 1/4$, we get a lower bound from (4.12) in the form

$$\begin{aligned} \mathcal{G}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) &\geq \left[\frac{1}{16} \eta \bar{\sigma}^2 |DF_1(x) - DF_{2,+}^M(t, x)|^2 - \varepsilon c \right] \\ &\geq \left[\frac{c^2 \eta}{4\bar{\sigma}^2} \left(z - \hat{x} e^{c(t-T)} \right)^2 - \varepsilon c \right]. \end{aligned}$$

Since this is less than the bound for Case II when both terms are negative, the conclusion of the lemma follows. ■

Theorem 4.6 *Assume $\delta = 2\varepsilon, \eta \in (\varepsilon/(\varepsilon + cH^2/\bar{\sigma}^2), 1/4)$, and that $z^2 c \eta \geq 8\varepsilon \bar{\sigma}^2$. Let \bar{u} be the control based on the function \bar{U}^δ defined in (4.4), i.e., $\bar{u}(t, x) = -\bar{\sigma} D\bar{U}^\delta(t, x)$. Then up to an exponentially negligible term, we have*

$$-\varepsilon \log Q^\varepsilon(0, 0; \bar{u}) \geq 2I_1(\varepsilon, \eta, T, \hat{x}, M)1_{\{T \geq t^*\}} + 2I_2(\varepsilon, T)1_{\{T < t^*\}},$$

where

$$I_1(\varepsilon, \eta, T, \hat{x}, M) = (1 - \eta)\bar{U}^\delta(0, 0) + \left(\log \left[\frac{1}{\hat{x}} \left(z - \sqrt{4\varepsilon \bar{\sigma}^2 / c \eta} \right) \right] \wedge 0 \right) \varepsilon.$$

$$I_2(\varepsilon, T) = 2L - cT\varepsilon$$

and $L = \frac{1}{2} \frac{c}{\bar{\sigma}^2} A^2$.

Remark 4.7 *Although the bound provided by Theorem 4.6 takes a complicated form, it is important to note that it does not degrade as $T \rightarrow \infty$, and this is also reflected in the simulation data. Also, as noted previously there are natural scalings under which $\eta \rightarrow 0$ and $M \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Using the bound from below given in Lemma 4.1 and the explicit form of $F_{2,\pm}^M(0, 0)$, we obtain*

$$\bar{U}^\delta(0, 0) \geq \frac{c}{\frac{2c}{M} + \bar{\sigma}^2 - \bar{\sigma}^2 e^{-2cT}} \hat{x}^2 + \left(2L - \frac{c}{\bar{\sigma}^2} \hat{x}^2 \right) - \delta \log 3$$

Hence, we obtain the rate of decay

$$2L + \frac{c\hat{x}^2}{\bar{\sigma}^2} \left[\frac{e^{-2cT}}{1 - e^{-2cT}} \right]$$

uniformly in T as $\varepsilon \rightarrow 0$.

Proof of Theorem 4.6. The starting point is the representation (3.1) [but rewritten for the more general process model and with time dependent u], which is valid for every $\varepsilon > 0$. We can restrict to v such that $\tau^\varepsilon \leq T$ w.p.1, obtaining

$$-\varepsilon \log Q^\varepsilon(0, 0; \bar{u}) = \inf_{v \in \mathcal{A}: \hat{\tau}^\varepsilon \leq T \text{ w.p.1}} \mathbb{E}_{0,0} \left[\frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} v(s)^2 ds - \int_0^{\hat{\tau}^\varepsilon} \bar{u}(s, \hat{X}^\varepsilon(s))^2 ds \right]. \quad (4.13)$$

We can also assume $T \geq t^*$ since the bound is straightforward otherwise. We recall that under (4.3) the subsolution property is preserved for $\bar{U}^\delta(t, x)$ at $T - t^*$, i.e., that

$$U^\delta(T - t^*, x) \leq F_1(x) \text{ for all } x \in [-A, A].$$

Next consider any control in the representation such that $\hat{\tau}^\varepsilon \leq T$ w.p.1. We will apply Itô's formula separately over the intervals $[0, (T - t^*) \wedge \hat{\tau}^\varepsilon]$ and $[(T - t^*) \wedge \hat{\tau}^\varepsilon, \hat{\tau}^\varepsilon]$ and also use the boundary condition $\bar{U}^{\delta, \eta}(t, \pm A) \leq F_1(\pm A) \leq 0$ for $t \in [0, T]$. Since $U^{\delta, \eta}(T - t^*, x) \leq F_1(x)$ we obtain

$$\begin{aligned} -\bar{U}^{\delta, \eta}(0, 0) &\geq \mathbb{E}_{0,0} \left[\bar{U}^{\delta, \eta}(\hat{\tau}^\varepsilon, \hat{X}^\varepsilon(\hat{\tau}^\varepsilon)) - \bar{U}^{\delta, \eta}((T - t^*) \wedge \hat{\tau}^\varepsilon, \hat{X}^\varepsilon((T - t^*) \wedge \hat{\tau}^\varepsilon)) \right] 1_{\{\hat{\tau}^\varepsilon \geq T - t^*\}} \\ &\quad + \mathbb{E}_{0,0} \left[\bar{U}^{\delta, \eta}((T - t^*) \wedge \hat{\tau}^\varepsilon, \hat{X}^\varepsilon((T - t^*) \wedge \hat{\tau}^\varepsilon)) - \bar{U}^{\delta, \eta}(0, 0) \right]. \end{aligned} \quad (4.14)$$

Using Lemma 6.1 and recalling the definition of $\mathcal{G}^\varepsilon[W, U]$ in (4.5), the contribution from $[0, (T - t^*) \wedge \hat{\tau}^\varepsilon]$ gives

$$\begin{aligned} &\mathbb{E}_{0,0} \left[\bar{U}^{\delta, \eta}((T - t^*) \wedge \hat{\tau}^\varepsilon, \hat{X}^\varepsilon((T - t^*) \wedge \hat{\tau}^\varepsilon)) - \bar{U}^{\delta, \eta}(0, 0) \right] \\ &= \mathbb{E}_{0,0} \int_0^{(T - t^*) \wedge \hat{\tau}^\varepsilon} \left[\mathcal{G}^\varepsilon[\bar{U}^{\delta, \eta}, \bar{U}^\delta](s, \hat{X}^\varepsilon(s)) ds - \frac{1}{4} v(s)^2 ds + \frac{1}{2} \bar{u}(s, \hat{X}^\varepsilon(s))^2 \right] ds. \end{aligned}$$

An analogous formula holds for $[(T - t^*) \wedge \hat{\tau}^\varepsilon, \hat{\tau}^\varepsilon]$, save that since $\bar{U}^{\delta, \eta} = \bar{U}^\delta = F_1$, the term $\mathcal{G}^\varepsilon[\bar{U}^{\delta, \eta}, \bar{U}^\delta]$ simplifies to $\mathcal{G}^\varepsilon[F_1]$. Rearranging and using (4.14),

$$\begin{aligned} \mathbb{E}_{0,0} \left[\frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} v(s)^2 ds - \int_0^{\hat{\tau}^\varepsilon} \bar{u}(s, \hat{X}^\varepsilon(s))^2 ds \right] &\geq 2\bar{U}^{\delta, \eta}(0, 0) \\ &\quad + \mathbb{E}_{0,0} \int_0^{(T - t^*) \wedge \hat{\tau}^\varepsilon} 2\mathcal{G}^\varepsilon[\bar{U}^{\delta, \eta}, \bar{U}^\delta](s, \hat{X}^\varepsilon(s)) ds + \mathbb{E}_{0,0} \int_{(T - t^*) \wedge \hat{\tau}^\varepsilon}^{\hat{\tau}^\varepsilon} 1_{\{\hat{\tau}^\varepsilon \geq T - t^*\}} 2\mathcal{G}^\varepsilon[F_1](s, \hat{X}^\varepsilon(s)) ds. \end{aligned}$$

We now replace each term by a lower bound, using Lemmas 4.3, 4.4 and 4.5 for $2\mathcal{G}^\varepsilon[\bar{U}^{\delta, \eta}, \bar{U}^\delta]$. Since the bounds are independent of the control process v , the representation (4.13) implies

$$-\varepsilon \log Q^\varepsilon(0, 0; \bar{u}) \geq 2(1 - \eta)\bar{U}^\delta(0, 0) + \int_J \left[\frac{c^2 \eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{c(s-T)} \right)^2 - 2\varepsilon c \right] ds - t^* 2\varepsilon c,$$

where J are the times in $[0, T - t^*]$ where the integrand is negative.

We next use the constraint $z^2 c \eta \geq 8\varepsilon \bar{\sigma}^2$, which guarantees that for $T - s$ sufficiently large the integrand is in fact positive. Let

$$\frac{c^2 \eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{-cb} \right)^2 - 2\varepsilon c \text{ or } b = -\frac{1}{c} \log \left[\frac{1}{\hat{x}} \left(z - \sqrt{4\varepsilon \bar{\sigma}^2 / c \eta} \right) \right].$$

Then since the integrand is only negative for $s \geq T - b$, we obtain the lower bound $-[(b - t^*) \vee 0] 2\varepsilon c$ for the integral. Adding the remaining $-t^* 2\varepsilon c$ then gives the result as stated. ■

4.4 Simulation results for the linear problem

In this subsection we present simulation data for the linear problem and make several comments on the application of the algorithm. For comparison purposes, we consider the same two sided problem corresponding to the data from Tables 1, 2, 5 and 7. Thus we consider the small noise diffusion process with drift $b(x) = -V'(x)$, where $V(x) = \frac{1}{2}x^2$, diffusion coefficient $\sqrt{\varepsilon}$, and starting from the stable equilibrium point $x = 0$. The goal is to estimate the probability of exiting the set $(-1, 1)$ by a given time T .

As discussed in Subsection 4.2, the change of measure for the importance sampling scheme is based on the subsolution (4.4). In order to apply it to a given pair (ε, T) , one needs to choose the parameters $(\hat{x}, M, t^*, \delta)$. Before presenting simulation data, we comment on these choices.

The analysis in Subsection 4.3 assumes $t^* \geq -\frac{2}{c} \log \frac{2c}{M\bar{\sigma}^2}$ and $\delta = 2\varepsilon$, and we will take $t^* = -\frac{2}{c} \log \frac{2c}{M\bar{\sigma}^2}$. As noted before Lemmas 4.3-4.5, it is natural to allow quantities such as z and H , which characterize the region where the solution to the LQR replaces the subsolution based on the quasipotential, to depend on ε . One would like the width of this region to scale like ε^κ , with $\kappa \in (0, 1/2]$, which in turn suggests that M scale like $2\hat{x}^2 c / \bar{\sigma}^2 \varepsilon^{2\kappa}$. However, the exponential negligibility of certain terms that holds when parameters such as z, H and M are independent of ε need not hold when they depend on ε . For example, the exponential negligibility of the term $(1 - \eta)\rho_1(t, x)\gamma_1$ appearing in Lemma 4.3 should be examined.

Recall that the exponential rate of decay of terms like $(1 - \eta)\rho_1(t, x)\gamma_1$ is bounded by the smallest value of $F_1(x) - F_{2,+}^M(t, x)$. A lower bound of the form $\hat{x}^2 c [1/\bar{\sigma}^2 - 1/(\bar{\sigma}^2 + c/M)]$ was obtained in the proof of Lemma 4.3. Inserting the given scaling and approximating for small ε gives $c\varepsilon^{2\kappa}/2\bar{\sigma}^2$, and upon dividing by $\delta = 2\varepsilon$ gives the exponent $c\varepsilon^{2\kappa-1}/4\bar{\sigma}^2$. Hence exponential negligibility requires $\kappa \in (0, 1/2)$, with smaller values of κ giving a faster rate of decay. Note however that the analysis assumes $M \geq 4c/\bar{\sigma}^2$ and $z^2 c \eta \geq 8\varepsilon\bar{\sigma}^2$. With regard to the condition $M \geq 4c/\bar{\sigma}^2$, inserting the given scaling for M we get the constraint $\hat{x}^2/2\varepsilon^{2\kappa} \geq 1$. This is clearly satisfied for small $\varepsilon > 0$ if \hat{x} is of order 1. One may also take here \hat{x} to be of order ε^λ and the constraint will be satisfied for small ε if $\lambda < \kappa$. We also remark here that for the nonlinear problem, the condition $M \geq 4c/\bar{\sigma}^2$ needs to be strengthened to $M > 4c/\bar{\sigma}^2$. With regard to the condition $z^2 c \eta \geq 8\varepsilon\bar{\sigma}^2$, inserting the given scaling for M and recalling the definition $z \doteq \hat{x}(c/M\bar{\sigma}^2)^{1/2}/2$, we obtain the constraint $\varepsilon^{2\kappa-1} \geq 64\bar{\sigma}^2/\eta c$. This constraint is satisfied for small enough ε when $\kappa \in (0, 1/2)$, and moreover one can allow $\eta \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Below we present simulation data for various choices of the parameters as indicated in the corresponding tables. In Table 8, estimated probabilities are reported when $M = 4$ and $\hat{x} = 1$, whereas the related relative error per sample estimates are reported in Table 9 (since the relative errors are consistently smaller we now round to the nearest 1/10). In Tables 10 and 11 relative errors per sample estimates are reported for combinations of (M, \hat{x}) that depend on ε . The related probability estimates are almost identical to those in Table 8. Note that the degradation in performance as T is gets larger observed previously, is no longer present. This agrees with the theoretical performance bound appearing in Theorem 4.6.

$\varepsilon \mid T$	1.5	2.5	5	7	10	14	18	23
0.20	$9.1e-03$	$2.3e-02$	$5.7e-02$	$8.3e-02$	$1.2e-01$	$1.7e-01$	$2.1e-01$	$2.7e-01$
0.16	$2.2e-03$	$6.6e-03$	$1.8e-02$	$2.7e-02$	$4.0e-02$	$5.7e-02$	$7.4e-02$	$9.5e-02$
0.13	$5.1e-04$	$1.6e-03$	$4.6e-03$	$6.9e-03$	$1.1e-02$	$1.5e-02$	$2.0e-02$	$2.6e-02$
0.11	$1.1e-04$	$3.9e-04$	$1.2e-03$	$1.8e-03$	$2.8e-03$	$4.1e-03$	$5.4e-03$	$7.0e-03$
0.09	$1.3e-05$	$5.2e-05$	$1.7e-04$	$2.6e-04$	$4.1e-04$	$5.9e-04$	$7.8e-04$	$1.0e-03$
0.07	$4.3e-07$	$2.2e-06$	$7.6e-06$	$1.2e-05$	$1.9e-05$	$2.8e-05$	$3.7e-05$	$4.8e-05$
0.05	$9.7e-10$	$6.9e-09$	$2.8e-08$	$4.4e-08$	$7.0e-08$	$1.1e-07$	$1.4e-07$	$1.8e-07$

Table 8: Estimated values for different pairs (ε, T) . $M = 4$, $\hat{x} = 1$.

$\varepsilon \mid T$	1.5	2.5	5	7	10	14	18	23
0.20	1.7	1.4	1.0	0.9	0.6	0.6	0.7	0.9
0.16	2.1	1.8	1.2	1.0	0.8	0.7	0.7	0.8
0.13	2.4	2.2	1.6	1.4	1.1	0.8	0.8	0.8
0.11	2.9	2.7	2.0	1.6	1.3	1.1	1.0	0.9
0.09	3.6	3.6	2.7	2.3	1.8	1.5	1.3	1.2
0.07	4.9	5.7	4.2	3.4	2.9	2.4	2.1	1.9
0.05	8.9	13.0	9.9	8.3	6.8	5.7	5.0	4.4

Table 9: Relative errors per sample for different pairs (ε, T) . $M = 4$ and $\hat{x} = 1$.

5 The non-linear one dimensional problem

In this section, we extend the construction of Section 4 to the general non-linear one dimensional setting. We also generalize the notation and allow the stable equilibrium to be an arbitrary point x_0 . Consider the process model (1.1) and assume that $b, \sigma \in \mathcal{C}^1(\mathbb{R})$ and that $b(x_0) = 0$, $b'(x_0) < 0$ and $\sigma^2(x) \geq \sigma_1^2 > 0$ for all $x \in \mathbb{R}$. Thus we can write $b(x) = -V'(x)$ with unique local minimum at $x = x_0$ and $V''(x_0) > 0$. It is easy to see that the quasipotential with respect to the equilibrium point x_0 takes the form

$$S(x_0, x) = \int_{x_0}^x -2 \frac{b(z)}{\sigma^2(z)} dz.$$

The problem of interest is to estimate the exit probability

$$\theta^\varepsilon = \mathbb{P}_{x_0} \{X^\varepsilon \text{ hits } A_1 \text{ or } A_2 \text{ before time } T\},$$

where x_0 is the initial (and rest) point such that $x_0 \in (A_1, A_2)$. Furthermore, we assume that $b(x) < 0$ for all $x \in (x_0, A_2]$ and $b(x) > 0$ for all $x \in [A_1, x_0)$. Set $L \doteq \frac{1}{2} [S(x_0, A_1) \vee S(x_0, A_2)]$.

The approach to the nonlinear problem is to merge the linearized dynamics around the equilibrium point with the subsolution based on the quasipotential. This subsolution is

$$\bar{F}_1(x) = 2L - S(x_0, x).$$

Observe that the second order approximation to this function around the equilibrium point x_0 is

$$F_1(x) = 2L - \frac{c}{\bar{\sigma}^2} (x - x_0)^2,$$

$\varepsilon \mid T$	2.5	5	7	10	14	18	23
0.20	1.3	0.9	0.7	0.6	0.6	0.7	1.0
0.16	1.5	1.1	0.8	0.7	0.7	0.7	0.9
0.13	1.7	1.2	1.0	0.8	0.7	0.7	0.8
0.11	1.8	1.4	1.2	0.9	0.8	0.7	0.8
0.09	2.0	1.6	1.3	1.1	0.9	0.8	0.8
0.07	2.2	1.9	1.6	1.3	1.1	1.0	0.9
0.05	2.4	2.5	2.1	1.7	1.5	1.3	1.1

Table 10: Relative errors per sample for different pairs (ε, T) . $M = \frac{2}{\sqrt{\varepsilon}}$ and $\hat{x} = 1$.

$\varepsilon \mid T$	2.5	5	7	10	14	18	23
0.20	1.5	0.9	0.7	0.7	0.9	1.1	1.3
0.16	1.7	1.0	0.8	0.7	0.8	1.0	1.2
0.13	1.8	1.1	0.8	0.8	0.8	1.0	1.0
0.11	1.9	1.1	0.9	0.7	0.9	0.9	1.1
0.09	2.2	1.2	0.9	0.8	0.8	0.9	1.1
0.07	2.4	1.3	1.0	0.8	0.8	0.9	1.1
0.05	2.9	1.5	1.1	0.9	0.8	0.9	1.0

Table 11: Relative errors per sample for different pairs (ε, T) . $M = \frac{2}{\varepsilon^{0.3}}$ and $\hat{x} = \varepsilon^{0.15}$.

where $c = -b'(x_0)$ and $\bar{\sigma} = \sigma(x_0)$. Let \hat{x}_+, \hat{x}_- be such that $\hat{x}_+ - x_0 = x_0 - \hat{x}_-$. The appropriate translated version of $F_{2,+}^M$ is

$$F_{2,+}^M(t, x) = a^M(t) \left((x - x_0) - (\hat{x}_+ - x_0)e^{-c(t-T)} \right)^2 + F_1(\hat{x}_+),$$

and $F_{2,-}^M(t, x_0 - x) = F_{2,+}^M(t, x_0 + x)$.

The subsolution for times less than $T - t^*$ will be the mollification of $F_{2,+}^M(t, x) \wedge F_{2,-}^M(t, x) \wedge \bar{F}_1(x)$. Note that since $\bar{F}_1(x)$ agrees with $F_1(x)$ up to second order it is still the case that $F_{2,+}^M(t, x) \wedge F_{2,-}^M(t, x)$ will be smallest near $x = x_0$. Letting

$$U^\delta(t, x) = -\delta \log \left(e^{-\frac{1}{\delta} F_{2,+}^M(t, x)} + e^{-\frac{1}{\delta} F_{2,-}^M(t, x)} + e^{-\frac{1}{\delta} \bar{F}_1(x)} \right) \quad (5.1)$$

and

$$\bar{U}^\delta(t, x) = \begin{cases} \bar{F}_1(x), & t > T - t^* \\ U^\delta(t, x), & t \leq T - t^* \end{cases}, \quad (5.2)$$

the suggested importance sampling control that is used for the simulation is

$$\bar{u}(t, x) = -\sigma(x) D\bar{U}^\delta(t, x).$$

Notice that this construction reduces to the construction of the linear case if the potential is indeed quadratic, since then $\bar{F}_1(x) = F_1(x)$.

In Subsection 5.1 we present simulation data for the nonlinear problem, demonstrating the effectiveness of the suggested change of measure. The analysis and the theoretical bound for the performance of this scheme are completely analogous to the linear problem, modulo the additional error coming from the linearization of the dynamics in the neighborhood of the stable equilibrium point. In Subsection 5.2 we rigorously analyze the performance of this algorithm.

5.1 Simulation results for nonlinear problem

In this subsection, we present simulation data for the nonlinear problem. We take the drift to be $b(x) = -V'(x)$, where the potential function is $V(x) = \frac{1}{2}(x^2 - 1)^2$. This potential function has two stable points at -1 and at $+1$, and an unstable equilibrium at 0 . We assume that the starting point is at the right equilibrium point $x_0 = -1$ and the exit set is the level set of the potential function $\mathcal{D} = \{x : V(x) \leq L\}$, with $L = 0.45$. Thus exit occurs from either of the points $A_1 = -1.40$ or $A_2 = -0.23$.

Notice that the local quadratic approximation around the equilibrium point is $V_q(x) = \frac{1}{2}c(x+1)^2$ with $c = 4$. Moreover, we have chosen, for simplicity, the diffusion coefficient to be constant $\sigma(x) = 1$. $N = 10^7$ independent trajectories were used for the simulations.

We first investigate the performance of a change of measure based on the quasipotential subsolution. Thus we change the measure via the control $\bar{u}(x) = -D\bar{F}_1(x)$. Estimated values and the corresponding estimated relative errors per sample for several values of (ε, T) are in Tables 12 and 13, respectively.

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10
0.14	$3.01e-03$	$8.21e-03$	$1.36e-02$	$2.48e-02$	$5.82e-02$	$6.22e-02$	$4.11e-02$	$4.29e-02$	$5.31e-02$
0.12	$1.03e-03$	$2.91e-03$	$4.92e-03$	$8.95e-03$	$1.46e-02$	$1.73e-02$	$2.10e-02$	$1.81e-02$	$1.72e-02$
0.09	$8.27e-05$	$2.52e-04$	$4.35e-04$	$8.10e-04$	$1.33e-03$	$1.53e-03$	$1.65e-03$	$1.58e-03$	$1.57e-03$
0.07	$4.60e-06$	$1.49e-05$	$2.64e-05$	$4.97e-05$	$7.63e-05$	$1.11e-04$	$9.74e-05$	$1.21e-04$	$9.19e-05$
0.05	$2.49e-08$	$8.91e-08$	$1.62e-07$	$3.15e-07$	$5.03e-07$	$6.19e-07$	$6.78e-07$	$5.85e-07$	$6.29e-07$
0.03	$1.25e-13$	$5.40e-13$	$1.03e-12$	$2.10e-12$	$3.80e-12$	$6.12e-12$	$4.87e-12$	$1.04e-11$	$5.18e-12$

Table 12: Using the subsolution based on quasipotential throughout. Estimated values for different pairs (ε, T) .

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10
0.14	2	2	4	21	799	953	127	169	488
0.12	2	2	4	18	125	315	649	368	301
0.09	2	2	4	20	143	155	311	173	288
0.07	2	2	4	17	68	433	192	540	272
0.05	2	2	4	16	77	296	410	148	287
0.03	2	2	3	14	160	638	347	1933	317

Table 13: Using subsolution based on quasipotential throughout. Relative errors per sample for different pairs (ε, T) .

As we see from Table 13, even though the quasipotential subsolution performs relatively well for small values of T , there is a clear degradation of performance as T gets larger. It is also interesting to note that the degradation is uniform across all values of ε for the same value of

T . This behavior parallels what was observed for the linear problem. As was mentioned there, the large per sample relative errors for $T \geq 2.5$ should not be taken as being accurate, but just indicative of poor performance.

Next we investigate how the suggested change of measure performs. To apply the control, we choose values for the parameters $(\hat{x}, M, t^*, \delta)$ according to the discussion in Subsection 4.4. However, for reasons that will become clearer in the proof of Lemma 5.6, we need to strengthen the condition $M \geq 4c/\bar{\sigma}^2$ to $M > 4c/\bar{\sigma}^2$, say $M \geq 5c/\bar{\sigma}^2$. When we link the other parameters to ε the $z \rightarrow 0$ as $\varepsilon \rightarrow 0$, and because of this we do not explicitly take into account the error from the approximation around the neighborhood of the rest point of the true dynamics by its linearization when selecting the parameters for the implementation of the scheme.

Estimated values and corresponding estimated relative errors of the exit probabilities of interest for different pairs (ε, T) and different combinations values for z are in Tables 14 through 18. Estimated relative errors for $M = \frac{2c}{\bar{\sigma}^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.4$ and $\hat{x} = 0.4$ are reported in Table 14, whereas the related relative errors are reported in Table 15. In Tables 16 and 17, we report only estimated relative errors for the same value of κ but for $\hat{x} = 0.5$ and $\hat{x} = 1$ respectively. The related probability estimates are almost identical to those of Table 14, so they are not repeated.

Note that for Table 16, $t^* \geq T$ when $T = 0.5$ and for $\varepsilon \leq 0.05$. Similarly, for Table 17, $t^* \geq T$ when $T = 0.5$ and for $T = 1$ when $\varepsilon \leq 0.09$. For such values, the quasipotential subsolution is being used everywhere (see (5.2)) and the numerical results for these values agree with those from Table 13.

In order to illustrate the effect when the linear approximation is used over a relatively large region, the data in Table 18 are estimated relative errors when M is considerably smaller than before, and thus z is considerably larger. In particular, we have taken $\kappa = 0.25$ and $\hat{x} = 1$. Comparing Tables 17 and 18, we notice that if the error from the linearization is not confined to a small enough region then the algorithm degrades in ε though it appears stable in T . This is consistent with the theoretical results, which imply a uniformity in T but only logarithmic optimality in ε . That said, one would like to minimize errors associated with linearization as far as possible. As noted in Subsection 4.4, one should choose the scaling parameter $\kappa \in (0, 1/2)$. However, with the nonlinear problem minimizing the region over which the approximation is used calls for larger κ , and so one want it close to but not equal to $1/2$. For the problems considered here, $\kappa = .4$ worked well.

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10	13
0.14	3.07e-03	8.36e-03	1.39e-02	2.49e-02	4.11e-02	5.12e-02	6.24e-02	8.33e-02	1.04e-01	1.34e-01
0.12	1.05e-03	2.97e-03	4.99e-03	9.06e-03	1.52e-02	1.91e-02	2.32e-02	3.12e-02	3.92e-02	5.09e-03
0.09	8.36e-05	2.53e-04	4.39e-04	8.12e-04	1.38e-03	1.76e-03	2.14e-03	2.89e-03	3.65e-03	4.78e-03
0.08	2.36e-05	7.41e-05	1.29e-04	2.41e-04	4.11e-04	5.24e-04	6.38e-04	8.63e-04	1.08e-03	1.43e-03
0.07	4.60e-06	1.49e-05	2.65e-05	5.01e-05	8.55e-05	1.09e-04	1.33e-04	1.81e-04	2.28e-04	2.99e-04
0.05	2.48e-08	8.91e-08	1.63e-07	3.16e-07	5.44e-07	6.99e-07	8.51e-07	1.16e-06	1.47e-06	1.92e-06
0.04	2.57e-10	9.89e-10	1.85e-09	3.64e-09	6.35e-09	8.15e-09	1.01e-08	1.36e-08	1.72e-08	2.26e-08
0.03	1.25e-13	5.38e-13	1.03e-12	2.08e-12	3.68e-12	4.74e-12	5.80e-12	7.94e-12	1.01e-11	1.32e-11

Table 14: Estimated probability values for different pairs (ε, T) using the exponential mollification. Parameter choices $M = \frac{2c}{\bar{\sigma}^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.4$ and $\hat{x} = 0.4$

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10	13
0.14	3.7	2.3	1.8	1.3	1.1	1.0	1.0	1.1	1.3	1.7
0.12	4.2	2.6	2.0	1.5	1.2	1.1	1.1	1.1	1.2	1.5
0.09	5.1	3.3	2.6	1.9	1.5	1.3	1.2	1.2	1.2	1.2
0.08	5.3	3.7	2.8	2.1	1.8	1.4	1.3	1.2	1.2	1.2
0.07	5.5	4.1	3.2	2.3	1.8	1.6	1.5	1.3	1.2	1.2
0.05	4.9	5.4	4.5	3.2	2.5	2.2	2.0	1.7	1.6	1.3
0.04	3.2	6.5	5.4	4.1	3.2	2.8	2.6	2.2	2.0	1.8
0.03	2.8	8.0	7.3	5.6	4.4	3.9	3.5	3.0	2.7	2.4

Table 15: Relative errors per sample for different pairs (ε, T) using the exponential mollification. Parameter choices $M = \frac{2c}{\sigma^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.4$ and $\hat{x} = 0.4$.

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10	13
0.14	2.7	2.5	2.0	1.5	1.2	1.0	1.0	1.0	1.0	1.1
0.12	2.6	2.9	2.4	1.8	1.4	1.2	1.1	1.1	1.1	1.1
0.09	2.1	3.7	3.1	2.4	1.9	1.7	1.6	1.4	1.3	1.2
0.08	1.9	4.1	3.5	2.7	2.1	1.8	1.8	1.5	1.4	1.3
0.07	1.9	4.5	3.9	3.0	2.4	2.1	1.9	1.7	1.5	1.4
0.05	1.8	5.5	5.2	4.2	3.2	2.9	2.7	2.3	2.0	1.9
0.04	1.7	6.0	6.4	5.2	4.1	3.7	3.4	2.9	2.7	2.3
0.03	2.0	5.9	8.2	6.9	5.6	4.9	4.5	3.9	3.6	3.1

Table 16: Relative errors per sample for different pairs (ε, T) using the exponential mollification. Parameter choices $M = \frac{2c}{\sigma^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.4$ and $\hat{x} = 0.5$.

5.2 Analysis of the simulation scheme for the nonlinear problem

In this subsection, we present the theoretical analysis of the simulation scheme for general one-dimensional non-linear dynamics and provide rigorous bounds on performance. As for the linear case, the analysis is valid for $\varepsilon > 0$ without degradation as $T \rightarrow \infty$. The analysis and the theoretical bound for the performance of this scheme are completely analogous to the linear problem, modulo the additional error coming from the linearization of the dynamics in the neighborhood of the stable equilibrium point.

To distinguish between the linear and the nonlinear problem we need to introduce some notation. For a function $W \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ we define the operator

$$\bar{\mathcal{G}}^\varepsilon[W](t, x) = W_t(t, x) + \bar{\mathbb{H}}(x, DW(t, x)) + \frac{\varepsilon}{2} \sigma^2(x) D^2 W(t, x), \quad \bar{\mathbb{H}}(x, p) = b(x)p - \frac{1}{2} |\sigma(x)p|^2.$$

In analogy to (4.5), for smooth functions W, U , we define

$$\bar{\mathcal{G}}^\varepsilon[W, U](t, x) = \bar{\mathcal{G}}^\varepsilon[W](t, x) - \frac{1}{2} |\sigma(x) (DW(t, x) - DU(t, x))|^2.$$

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10	13
0.14	1.5	2.4	2.6	2.2	1.8	1.6	1.5	1.3	1.2	1.0
0.12	1.5	2.3	3.0	2.7	2.2	2.0	1.9	1.6	1.4	1.3
0.09	1.6	2.2	3.9	3.8	3.2	2.9	2.7	2.4	2.1	1.9
0.08	1.6	2.1	4.1	4.2	3.7	3.3	3.1	2.7	2.5	2.1
0.07	1.8	2.1	4.2	4.7	4.0	3.7	3.4	3.0	2.7	2.4
0.05	1.8	2.1	4.4	6.0	5.3	4.8	4.5	3.9	3.6	3.2
0.04	1.9	2.1	4.8	7.1	6.3	5.8	5.5	4.8	4.3	3.9
0.03	2.0	2.1	3.5	8.7	8.2	7.5	6.9	6.2	5.6	4.9

Table 17: Relative errors per sample for different pairs (ε, T) using the exponential mollification. Parameter choices $M = \frac{2c}{\bar{\sigma}^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.4$ and $\hat{x} = 1$.

$\varepsilon \mid T$	0.5	1	1.5	2.5	4	5	6	8	10	13
0.14	1.5	5	5	4	3	3	3	2	2	2
0.12	1.5	7	7	6	5	5	4	4	3	3
0.09	1.6	14	19	17	14	13	12	11	9	8
0.08	1.6	21	30	28	23	22	20	17	15	14
0.07	1.8	31	54	51	42	39	36	31	28	25
0.05	1.8	36	276	278	214	220	185	153	148	128
0.04	1.9	5	568	577	665	616	507	400	415	374
0.03	2.0	3	302	60	190	39	1878	1485	96	1562

Table 18: Relative errors per sample for different pairs (ε, T) using the exponential mollification. Parameter choices $M = \frac{2c}{\bar{\sigma}^2} \frac{\hat{x}^2}{\varepsilon^{2\kappa}}$ with $\kappa = 0.25$ and $\hat{x} = 1$.

Moreover, setting $c = -b'(x_0) > 0$ and $\bar{\sigma} = \sigma(x_0)$ we recall that

$$\mathcal{G}^\varepsilon[W](t, x) = W_t(t, x) + \mathbb{H}(x, DW(t, x)) + \frac{\varepsilon}{2} \bar{\sigma}^2 D^2 W(t, x), \quad \mathbb{H}(x, p) = -cxp - \frac{1}{2} |\bar{\sigma}p|^2.$$

The operators with bars correspond to the nonlinear problem, whereas the operators without bars give the corresponding first order approximations. We define an operator measuring the error from the approximation by

$$R^\varepsilon[W](t, x) = \bar{\mathcal{G}}^\varepsilon[W](t, x) - \mathcal{G}^\varepsilon[W](t, x) \tag{5.3}$$

Moreover, since $b(x)$ and $\sigma(x)$ are $\mathcal{C}^1(\mathbb{R})$, we can write for any $x \in \mathbb{R}$

$$\begin{aligned} b(x) &= b(x_0) + b'(x_0)(x - x_0) + R_1(x), \\ \sigma(x) &= \sigma(x_0) + R_2(x), \end{aligned}$$

where $R_1(x)|x|^{-2}$ and $R_2(x)|x|^{-1}$ are locally bounded. By assumption we have that $b(x_0) = 0$ and $\sigma^2(x) > 0$.

As with the linear problem, the subsolution used for the analysis is based on the δ -exponential mollification (5.1) reduced by the multiplicative factor $(1 - \eta)$. We recall that this differs from the subsolution used for the design, which has $\eta = 0$.

Next we proceed with the mathematical analysis of the scheme. The analysis is parallel to what was done for the linear problem, modulo adjustments due to the linearization of the dynamics in the neighborhood of the stable point, and therefore we mainly focus on the differences. In order to simplify the notation we assume without loss of generality (as it was done in the linear problem) that the stable equilibrium is $x_0 = 0$. We write $\hat{x}_+ = -\hat{x}_- = \hat{x}$, and for notational convenience assume that $A_2 = -A_1 = A$. As in the linear problem, $z = \hat{x}(c/M\bar{\sigma}^2)^{1/2}/2$ and $H = 10z$.

The following lemma bounds the error from the approximation.

Lemma 5.1 *Consider $(t, x) \in [0, T - t^*] \times [0, z]$. Then, for $z < \min\{1, A\}$*

$$\begin{aligned} |R^\varepsilon[F_2^M](t, x)| &\leq 2a^M(t)(z + \hat{x}e^{-c(t-T)}) \sup_{x \in [0, z]} |b(x) + cx| \\ &\quad + \left[2 \left(a^M(t)(z + \hat{x}e^{-c(t-T)}) \right)^2 + \varepsilon a^M(t) \right] \sup_{x \in [0, z]} |\sigma^2(x) - \bar{\sigma}^2| \\ &\leq C_0 \left\{ a^M(t)(z + \hat{x}e^{-c(t-T)})z^2 + \left(a^M(t)(z + \hat{x}e^{-c(t-T)}) \right)^2 z + a^M(t)\varepsilon z \right\}, \end{aligned}$$

where

$$C_0 = \sup_{x \in [0, A]} \left[\frac{|R_1(x)|}{|x|^2} + \frac{|R_2(x)|}{|x|} |2\bar{\sigma} + R_2(x)| \right].$$

In addition

$$\sup_{x \in [0, z]} |\bar{F}_1(x) - F_1(x)| \leq C_1 z^3,$$

where C_1 is a fixed constant.

Proof. Since by assumption $b(0) = 0$ and $\sigma^2(x) > 0$, for $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{aligned} R^\varepsilon[W](t, x) &= \bar{\mathbb{H}}(x, DW(t, x)) - \mathbb{H}(x, DW(t, x)) + \frac{\varepsilon}{2}(\sigma^2(x) - \bar{\sigma}^2)D^2W(t, x) \\ &= R_1(x)DW(t, x) - \frac{1}{2} \left(|\sigma(x)DW(t, x)|^2 - |\bar{\sigma}DW(t, x)|^2 \right) \\ &\quad + \frac{\varepsilon}{2}(\sigma^2(x) - \bar{\sigma}^2)D^2W(t, x) \\ &= R_1(x)DW(t, x) - \frac{1}{2}R_2(x)(2\bar{\sigma} + R_2(x))|DW(t, x)|^2 \\ &\quad + \frac{\varepsilon}{2}R_2(x)(2\bar{\sigma} + R_2(x))D^2W(t, x). \end{aligned}$$

For all $x \in [0, A]$, we have

$$|R_1(x)| \leq C|x|^2 \text{ and } |R_2(x)| \leq C|x| \tag{5.4}$$

for a constant C that depends on A . The bound follows by setting $W(t, x) = F_2^M(t, x)$ and using $DF_2^M(t, x) = 2a^M(t)(x - \hat{x}e^{-c(t-T)})$, and the proof of the first part of the lemma is concluded.

The proof of the second part goes as follows. Observe that

$$\begin{aligned}
\bar{F}_1(x) &= 2L - S(0, x) \\
&= 2L + 2 \int_0^x \frac{b(y)}{\sigma^2(y)} dy \\
&= 2L + 2 \int_0^x \frac{-cy + R_1(y)}{(\bar{\sigma} + R_2(y))^2} dy \\
&= F_1(x) + 2 \int_0^x \frac{R_1(y)}{(\bar{\sigma} + R_2(y))^2} dy + 2 \int_0^x \frac{c(2\bar{\sigma} + R_2(y))}{\bar{\sigma}^2(\bar{\sigma} + R_2(y))^2} R_2(y) y dy.
\end{aligned}$$

The last display and the bounds in (5.4) imply that

$$\sup_{x \in [0, z]} |\bar{F}_1(x) - F_1(x)| \leq C_1 z^3$$

which concludes the proof of the lemma. ■

For notational convenience we identify the quantity appearing in the upper bound for $|R^\varepsilon[F_2^M](t, x)|$ as given in Lemma 5.1:

$$r(\varepsilon, \hat{x}, M, t) = a^M(t)(z + \hat{x}e^{-c(t-T)})z^2 + \left(a^M(t)(z + \hat{x}e^{-c(t-T)})\right)^2 z + a^M(t)\varepsilon z. \quad (5.5)$$

The following lemma shows that the error term induced by the local approximation of the dynamics in the neighborhood of the stable equilibrium point does not degrade as T gets large.

Lemma 5.2 *We have that*

$$\int_0^{T-t^*} r(\varepsilon, \hat{x}, M, t) dt = J_1(t^*, T, M)z^3 + J_2(t^*, T, M)z^2\hat{x} + J_3(t^*, T, M)z\hat{x}^2 + J_4(t^*, T, M)\varepsilon z$$

where, letting $K = \frac{2c}{M} + \bar{\sigma}^2$,

$$\begin{aligned}
J_1(t^*, T, M) &= \frac{1}{2\bar{\sigma}^2} \left(1 - \frac{c}{\bar{\sigma}^2}\right) \log \frac{K - \bar{\sigma}^2 e^{-2cT}}{K - \bar{\sigma}^2 e^{-2ct^*}} + \frac{c}{2\bar{\sigma}^4} \left[\frac{K}{K - \bar{\sigma}^2 e^{-2ct^*}} - \frac{K}{K - \bar{\sigma}^2 e^{-2cT}} \right] \\
J_2(t^*, T, M) &= \frac{1}{\bar{\sigma}\sqrt{K}} \left(1 - \frac{c}{\bar{\sigma}^2}\right) \left[\log \frac{1 + \frac{\bar{\sigma}}{\sqrt{K}} e^{-ct^*}}{1 - \frac{\bar{\sigma}}{\sqrt{K}} e^{-ct^*}} - \log \frac{1 + \frac{\bar{\sigma}}{\sqrt{K}} e^{-cT}}{1 - \frac{\bar{\sigma}}{\sqrt{K}} e^{-cT}} \right] \\
&\quad + \frac{c}{2\bar{\sigma}^4} \left[\frac{2\bar{\sigma}^2 e^{-ct^*}}{K - \bar{\sigma}^2 e^{-2ct^*}} - \frac{2\bar{\sigma}^2 e^{-cT}}{K - \bar{\sigma}^2 e^{-2cT}} \right] \\
J_3(t^*, T, M) &= \frac{c}{2\bar{\sigma}^4} \left[\frac{\bar{\sigma}^2}{K - \bar{\sigma}^2 e^{-2ct^*}} - \frac{\bar{\sigma}^2}{K - \bar{\sigma}^2 e^{-2cT}} \right] \\
J_4(t^*, T, M) &= \frac{1}{2\bar{\sigma}^2} \log \frac{K - \bar{\sigma}^2 e^{-2cT}}{K - \bar{\sigma}^2 e^{-2ct^*}}.
\end{aligned}$$

In particular, $\lim_{T \rightarrow \infty} \int_0^{T-t^*} r(\varepsilon, \hat{x}, M, t) dt < \infty$.

The proof of this lemma follows by straightforward integration of $r(\varepsilon, \hat{x}, M, t)$. Moreover, we note that

$$J_i(t^*, T, M) = O(1) \text{ as } M \rightarrow \infty \text{ for all } i = 1, 2, 3, 4$$

and that the definition of $z = \hat{x}(c/M\bar{\sigma}^2)^{1/2}/2$ implies

$$\int_0^{T-t^*} r(\varepsilon, \hat{x}, M, t) dt = O(z^3 + z^2\hat{x} + z\hat{x}^2 + \varepsilon z), \text{ uniformly in } T < \infty, \text{ as } M \rightarrow \infty.$$

Remark 5.3 The following three lemmas are analogous to Lemmas 4.3-4.5 from the linear case. The important difference between the nonlinear and the linear case is that the statements involve approximation errors and the statements hold if one confines the linearized dynamics to a small neighborhood of the equilibrium point as dictated by the sizes of t^* and M , or equivalently by t^* and z . Due to the natural scaling of M in terms of ε as indicated in Subsection 4.4, as ε gets smaller, z will get smaller and be confined to a sufficiently small region that the statements of the lemmas are valid. However, the lemmas below are stated for z sufficiently small, without referencing to the natural scaling used in the simulation algorithm.

Lemma 5.4 *Assume that $(t, x) \in [0, T - t^*] \times [0, z]$, $\delta \geq \varepsilon$ and $\eta \leq 1/2$. Then, for sufficiently small z we have up to an exponentially negligible term*

$$\bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq -(1 - \eta)C_0 r(\varepsilon, \hat{x}, M, t).$$

Proof. As in the proof of Lemma 4.3 we need to show that in this region $\bar{F}_1(x) - F_{2,+}^M(t, x)$ is bounded by below away from zero. We have

$$\bar{F}_1(x) - F_{2,+}^M(t, x) = \hat{x}^2 \frac{c}{\bar{\sigma}^2} - \int_0^x \frac{2b(y)}{\sigma^2(y)} dy - a^M(t) \left(x - \hat{x} e^{-c(t-T)} \right)^2.$$

At $x = 0$, the value is minimized at $t = T - t^*$ and, as in Lemma 4.3, we obtain the lower bound $\hat{x}^2 c [1/\sigma^2 - 1/[\bar{\sigma}^2 + c/M]] > 0$. For $x > 0$ we use the decomposition

$$\bar{F}_1(x) - F_{2,+}^M(t, x) = [F_1(x) - F_{2,+}^M(t, x)] + [\bar{F}_1(x) - F_1(x)]. \quad (5.6)$$

Using the inequality $e^{-2ct^*} \leq [2c/M\bar{\sigma}^2]^4 \leq c/M\bar{\sigma}^2$, the definition of z and equation (4.8), we obtain that for all $(t, x) \in [0, T - t^*] \times [0, z]$

$$\begin{aligned} F_1(x) - F_{2,+}^M(t, x) &\geq \frac{c}{\bar{\sigma}^2} [\hat{x}^2 - z^2] - \frac{c}{K - \frac{c}{M}} \left(ze^{-ct^*} - \hat{x} \right)^2 \\ &\geq \frac{c}{\bar{\sigma}^2} \left[\frac{\hat{x}^2}{M} \frac{c}{\bar{\sigma}^2 + c/M} - z^2 \right] - z^2 \frac{ce^{-2ct^*}}{\bar{\sigma}^2 + c/M} \\ &\geq \frac{c}{\bar{\sigma}^2} z^2 \left[\frac{3\bar{\sigma}^2 - c/M}{\bar{\sigma}^2 + c/M} - \frac{c/M}{\bar{\sigma}^2 + c/M} \right] \\ &\geq \frac{c}{\bar{\sigma}^2} z^2 \left[\frac{2\bar{\sigma}^2 + 3c/M}{\bar{\sigma}^2 + c/M} - \frac{c/M}{\bar{\sigma}^2 + c/M} \right] \\ &= 2 \frac{c}{\bar{\sigma}^2} z^2. \end{aligned}$$

Hence, this term is of order z^2 . Moreover, by Lemma 5.1, we also have that

$$\sup_{x \in [0, z]} |\bar{F}_1(x) - F_1(x)| \leq C_1 z^3.$$

Thus, for sufficiently small z the first term on the right hand side of (5.6) dominates the second term. Hence the difference $\bar{F}_1(x) - F_{2,+}^M(t, x)$ is bounded from below away from zero if z is sufficiently small, and the term involving the weight ρ_1 is exponentially negligible.

We next use that $\beta_0(t, x) \geq 0$, $\gamma_2^M(t) > 0$ and $\eta \leq 1/2$, Lemma 5.1, and also (5.3) and (5.5). These imply that up to an exponentially negligible term,

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) &\geq (1 - \eta)\bar{\mathcal{G}}^\varepsilon[F_2^M](t, x) \\ &= (1 - \eta)\gamma_2^M(t) + (1 - \eta)R^\varepsilon[F_2^M](t, x) \\ &\geq -(1 - \eta)C_0 r(\varepsilon, \hat{x}, M, t), \end{aligned}$$

which concludes the proof of the lemma. ■

Lemma 5.5 *Assume that $(t, x) \in [0, T - t^*] \times [H, A]$ and assume $\delta \geq \varepsilon$. Define*

$$\eta_0(\varepsilon) \doteq \sup_{x \in [-A, -H] \cup [H, A]} \frac{-\varepsilon \sigma^2(x) D(b(x) \sigma^{-2}(x))}{-\varepsilon \sigma^2(x) D(b(x) \sigma^{-2}(x)) + b^2(x) \sigma^{-2}(x)}$$

Let $\varepsilon > 0$ be sufficiently small such that $\eta_0(\varepsilon) < 1/4$ and consider $\eta \in (\eta_0(\varepsilon), 1/4)$. Then, for sufficiently small z and up to exponentially negligible terms,

$$\bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq 0.$$

Proof. As with the linear problem we need to argue that there is a constant $c_2 > 0$ such that $F_{2,+}^M(t, x) - \bar{F}_1(x) \geq c_2 > 0$ for all $(t, x) \in [0, T - t^*] \times [H, A]$, which will then imply that the terms involving $\rho_2^{M,+}(t, x)$ are exponentially negligible. We have the decomposition

$$F_{2,+}^M(t, x) - \bar{F}_1(x) = [F_{2,+}^M(t, x) - F_1(H)] + [F_1(H) - \bar{F}_1(H)] + [\bar{F}_1(H) - \bar{F}_1(x)]$$

Focusing on $x = H$, Lemma 4.4 guarantees that $[F_{2,+}^M(t, H) - F_1(H)] > 0$ uniformly in $t \in [0, T - t^*]$. Moreover, a straightforward computation shows that for H sufficiently small, the lower bound of $[F_{2,+}^M(t, H) - F_1(H)]$ is of the order H^2 or equivalently z^2 (since $H = 10z$). By Lemma 5.1, the second term $[F_1(H) - \bar{F}_1(H)]$ is of order H^3 . Thus, for sufficiently small H , or equivalently for sufficiently small z , the term $F_{2,+}^M(t, H) - \bar{F}_1(H)$ is bounded by below away from zero. This statement, convexity of $x \mapsto F_{2,+}^M(t, x)$ and the fact that the third term of the last display is nonnegative for all $x \in [H, A]$, imply that there is a $c_2 > 0$ such that $F_{2,+}^M(t, x) - \bar{F}_1(x) \geq c_2$ for all $(t, x) \in [0, T - t^*] \times [H, A]$.

Hence, terms involving $\rho_2^{M,\pm}(t, x)$ are exponentially negligible. Since $\beta_0(t, x) \geq 0$, up to an exponentially negligible term

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) &\geq (1 - \eta)\rho_1(t, x)\bar{\mathcal{G}}^\varepsilon[\bar{F}_1](x) + \frac{1}{2}(\eta - 2\eta^2) |\sigma(x)\rho_1(t, x)D\bar{F}_1(x)|^2 \\ &= \varepsilon(1 - \eta)\sigma^2(x)D(b(x)\sigma^{-2}(x)) + 2\eta(1 - 2\eta)b^2(x)\sigma^{-2}(x). \end{aligned}$$

Thus for ε small enough that

$$\eta \in \left(\eta_0(\varepsilon), \frac{1}{4} \right),$$

we have, up to an exponentially negligible term

$$\bar{\mathcal{G}}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq 0,$$

concluding the proof of the lemma. ■

Lemma 5.6 *Assume that $(t, x) \in [0, T - t^*] \times [z, H]$ and that $M \geq 5c/\bar{\sigma}^2$. Set $\delta = 2\varepsilon$ and $\sigma_*^2 = \sup_{x \in [-A, A]} \sigma^2(x)$. Then, for sufficiently small z we have up to an exponentially negligible term*

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq & \frac{\sigma_*^2}{2} \left[\frac{1}{\bar{\sigma}^2} \left(\frac{c^2 \eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{c(t-T)} \right)^2 - 2\varepsilon c \right) + \Gamma(t, z, H, \hat{x}, \varepsilon, \eta, T) \right] \wedge 0 \\ & - C_0(1 - \eta)r(\varepsilon, \hat{x}, M, t). \end{aligned}$$

where

$$\begin{aligned} \Gamma(t, z, H, \hat{x}, \varepsilon, \eta, T) = & \inf_{x \in [z, H]} \left[\frac{c\eta}{2\bar{\sigma}^2} (D\bar{F}_1(x) - DF_1(x)) \left(z - \hat{x}e^{c(t-T)} \right) \right. \\ & \left. + \frac{1}{8}\eta (D\bar{F}_1(x) - DF_1(x))^2 + 2\varepsilon \left(\frac{c}{\bar{\sigma}^2} + D \left(\frac{b(x)}{\sigma^2(x)} \right) \right) \right] \end{aligned} \quad (5.7)$$

and $\sigma^2(x) \geq \sigma_1^2 > 0$ for all $x \in \mathbb{R}$.

Proof. As with the analysis of the linear problem, this region presents difficulties in its analysis, since the term $\bar{F}_1(x) - F_{2,+}^M(t, x)$ can be either positive or negative. This means that both $\rho_2^{M,+}$ and ρ_1 may be important. As with the linear problem, we ignore terms related to $\rho_2^{M,-}$ due to exponential negligibility.

Up to an exponentially negligible term

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) \geq & (1 - \eta) \frac{1}{4} \beta_0(t, x) + (1 - \eta) \rho_2^{M,+}(t, x) \bar{\mathcal{G}}^\varepsilon[F_{2,+}^M](t, x) + (1 - \eta) \rho_1(t, x) \bar{\mathcal{G}}^\varepsilon[\bar{F}_1](x) \\ & + \frac{1}{2} (\eta - 2\eta^2) \sigma^2(x) \left| \rho_2^{M,+}(t, x) DF_{2,+}^M(t, x) + \rho_1(t, x) D\bar{F}_1(x) \right|^2. \end{aligned}$$

We distinguish two cases depending on whether $\rho_1(t, x) > 1/2$ or $\rho_1(t, x) \leq 1/2$.

Case I: $\rho_1(t, x) > 1/2$. We know that $\beta_0(t, x) \geq 0$ and due to the positivity of $\gamma_{2,+}^M(t, x)$ we have by Lemma 5.1

$$\bar{\mathcal{G}}^\varepsilon[F_{2,+}^M](t, x) = \mathcal{G}^\varepsilon[F_{2,+}^M](t, x) + R^\varepsilon[F_{2,+}^M](t, x) \geq -C_0 r(\varepsilon, \hat{x}, M, t).$$

Next we need to argue that for $(t, x) \in [0, T - t^*] \times [z, H]$

$$DF_{2,+}^M(t, x)(D\bar{F}_1(x) - DF_{2,+}^M(t, x)) \geq 0.$$

Lemma 4.5 implies that

$$\begin{aligned} D\bar{F}_1(x) - DF_{2,+}^M(t, x) &= [D\bar{F}_1(x) - DF_1(x)] + [DF_1(x) - DF_{2,+}^M(t, x)] \\ &\leq [D\bar{F}_1(x) - DF_1(x)] + \left[\hat{x} \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\left(\frac{2c}{M\bar{\sigma}^2} \right)^2 - \frac{1}{2} \left(\frac{c}{M\bar{\sigma}^2} \right)^{1/2} \right) \right]. \end{aligned} \quad (5.8)$$

It follows easily from the proof of Lemma 5.1 that

$$\sup_{x \in [z, H]} |D\bar{F}_1(x) - DF_1(x)| \leq C_1 H^2 = 100C_1 z^2$$

for some constant C_1 which is independent of z . For the second term on the right hand side of (5.8) we have

$$\begin{aligned} &\hat{x} \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\left(\frac{2c}{M\bar{\sigma}^2} \right)^2 - \frac{1}{2} \left(\frac{c}{M\bar{\sigma}^2} \right)^{1/2} \right) \\ &\leq \hat{x} \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \frac{1}{2} \sqrt{\frac{c}{M\bar{\sigma}^2}} \left(8 \left(\frac{c}{M\bar{\sigma}^2} \right)^{3/2} - 1 \right) \\ &\leq z \frac{2c}{\bar{\sigma}^2} \frac{K}{K - \bar{\sigma}^2 e^{2c(t-T)}} \left(\frac{8}{5^{3/2}} - 1 \right) \\ &< 0 \end{aligned}$$

where we used the definition $z = \hat{x} \sqrt{c/(M\bar{\sigma}^2)}/2$ and the assumption $5c/(M\bar{\sigma}^2) \leq 1$. Thus, this term is strictly negative and of order z for z sufficiently small.

Thus for z sufficiently small, the first term on the right hand side of (5.8) is dominated by the second term which is negative. Together, with the non-positivity of $DF_{2,+}^M(t, x)$, we get that $DF_{2,+}^M(t, x)(D\bar{F}_1(x) - DF_{2,+}^M(t, x)) \geq 0$ for z sufficiently small.

Since $\eta \leq 1/4$, we obtain

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) &\geq (1 - \eta)\rho_1(t, x)\bar{\mathcal{G}}^\varepsilon[\bar{F}_1](x) + \frac{1}{16}\eta\sigma^2(x) |D\bar{F}_1(x) - DF_{2,+}^M(t, x)|^2 \\ &\quad - (1 - \eta)C_0 r(\varepsilon, \hat{x}, M, t) \\ &\geq \frac{\sigma^2(x)}{2} \left[\frac{1}{8}\eta (D\bar{F}_1(x) - DF_{2,+}^M(t, x))^2 + 2\varepsilon D \left(\frac{b(x)}{\sigma^2(x)} \right) \right] \\ &\quad - (1 - \eta)C_0 r(\varepsilon, \hat{x}, M, t). \end{aligned}$$

Case II: $\rho_1(t, x) \leq 1/2$. Similarly to the linear problem we have

$$\bar{\mathcal{G}}^\varepsilon[U^{\delta, \eta}, U^\delta](t, x) \geq (1 - \eta)\rho_1(t, x)\sigma^2(x) \left[\frac{1}{8} |D\bar{F}_1(x) - DF_{2,+}^M(t, x)|^2 + \varepsilon D \left(\frac{b(x)}{\sigma^2(x)} \right) \right]$$

Writing

$$D\bar{F}_1(x) - DF_{2,+}^M(t, x) = D\bar{F}_1(x) - DF_1(x) + DF_1(x) - DF_{2,+}^M(t, x)$$

and using, as in the linear problem, the estimate

$$DF_{2,+}^M(t, x) - DF_1(x) \geq \frac{2c}{\bar{\sigma}^2} \left(z - \hat{x}e^{c(t-T)} \right)$$

we obtain as in the proof of Lemma 4.5 that

$$\begin{aligned} \bar{\mathcal{G}}^\varepsilon[U^{\delta,\eta}, U^\delta](t, x) &\geq \left\{ \frac{\sigma^2(x)}{2} \left[\frac{1}{8}\eta \left(D\bar{F}_1(x) - DF_1(x) + \frac{2c}{\bar{\sigma}^2} \left(z - \hat{x}e^{c(t-T)} \right) \right)^2 + 2\varepsilon D \left(\frac{b(x)}{\sigma^2(x)} \right) \right] \right\} \wedge 0 \\ &\quad - C_0(1-\eta)r(\varepsilon, \hat{x}, M, t) \\ &\geq \frac{\sigma_*^2}{2} \left[\frac{1}{\bar{\sigma}^2} \left(\frac{c^2\eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{c(t-T)} \right)^2 - 2\varepsilon c \right) + \Gamma(t, z, H, \hat{x}, \varepsilon, \eta, T) \right] \wedge 0 \\ &\quad - C_0(1-\eta)r(\varepsilon, \hat{x}, M, t). \end{aligned}$$

where $\Gamma(t, z, H, \hat{x}, \varepsilon, \eta, T)$ was defined in (5.7). This concludes the proof of the lemma. ■

The performance bound is then summarized in the following theorem. The proof of Theorem 5.7 is the same as the proof of Theorem 4.6 for the linear case. So, it will not be repeated here.

Theorem 5.7 *Assume $\delta = 2\varepsilon$, $\eta \in (\eta_0(\varepsilon), 1/4)$, $z^2c\eta \geq 8\varepsilon\bar{\sigma}^2$ and that $M \geq 5c/\bar{\sigma}^2$, where $\eta_0(\varepsilon)$ is as in Lemma 5.5. Set $\sigma_*^2 = \sup_{x \in [-A, A]} \sigma^2(x)$. Let \bar{u} be the control based on the function \bar{U}^δ defined via (5.2), i.e., $\bar{u}(t, x) = -\sigma(x)D\bar{U}^\delta(t, x)$. Then, up to an exponentially negligible term, for $\varepsilon \in (0, \varepsilon_0)$ such that $\eta_0(\varepsilon_0) = 1/4$ and for z sufficiently small, we have*

$$\begin{aligned} -\varepsilon \log Q^\varepsilon(0, 0; \bar{u}) &\geq 2 \left[I_1(\varepsilon, \eta, T, t^*, \hat{x}, M) - (1-\eta)C_0 \int_0^{T-t^*} r(\varepsilon, \hat{x}, M, t) dt \right] 1_{\{T \geq t^*\}} \\ &\quad + 2I_2(\varepsilon, T) 1_{\{T < t^*\}}, \end{aligned}$$

where

$$\begin{aligned} I_1(\varepsilon, \eta, T, t^*, \hat{x}, M) &= (1-\eta)\bar{U}^\delta(0, 0) \\ &\quad + \frac{\sigma_*^2}{2} \int_J \left[\frac{1}{\bar{\sigma}^2} \left(\frac{c^2\eta}{2\bar{\sigma}^2} \left(z - \hat{x}e^{c(s-T)} \right)^2 - 2\varepsilon c \right) + \Gamma(s, z, H, \hat{x}, \varepsilon, \eta, T) \right] ds - t^*c^*\varepsilon, \end{aligned}$$

where J are the times in $[0, T - t^*]$ where the integrand is negative, $\Gamma(s, z, H, \hat{x}, \varepsilon, \eta, T)$ is as in (5.7),

$$\bar{U}^\delta(0, 0) \geq \frac{c}{K - \bar{\sigma}^2 e^{-2cT}} \hat{x}^2 + \left(2L - \frac{c}{\bar{\sigma}^2} \hat{x}^2 \right) - \delta \log 3$$

and

$$I_2(\varepsilon, T) = 2L - c^*T\varepsilon, \quad \text{and} \quad c^* = \sup_{x \in [-A, A]} \sigma^2(x) |D(b(x)/\sigma^2(x))| > 0.$$

The bound of Theorem 5.7 takes a complicated form, but as in the linear case, the performance does not degrade as $T \rightarrow \infty$. This was also reflected by the simulation data in Subsection 5.1. Let us now justify this claim.

Notice that, by Lemma 5.2 the term $\int_0^{T-t^*} r(\varepsilon, \hat{x}, M, t) dt$ is finite, uniformly in T . Next we need to argue, similarly to the linear problem, that when $T - s$ is sufficiently large and z is sufficiently small, the integrand of the second term in the definition of $I_1(\varepsilon, \eta, T, t^*, \hat{x}, M)$ is in fact positive. Let us denote the integrand of the second term by

$$B(s, z, H, \hat{x}, \varepsilon, \eta, T) + \Gamma(s, z, H, \hat{x}, \varepsilon, \eta, T)$$

where $B(s, z, H, \hat{x}, \varepsilon, \eta, T) = \frac{1}{\bar{\sigma}^2} \left(\frac{c^2 \eta}{2\bar{\sigma}^2} (z - \hat{x} e^{c(s-T)})^2 - 2\varepsilon c \right)$. The term $\Gamma(s, z, H, \hat{x}, \varepsilon, \eta, T)$ is composed by three terms and we shall argue below they are dominated (even when they are negative), by the second term in the definition $B(s, z, H, \hat{x}, \varepsilon, \eta, T)$, i.e., by $2\varepsilon c / \bar{\sigma}^2$ when z is small enough. This means, as in the case of the linear problem, that when the integral will be finite uniformly in T . Let us now support the claim just made. It is easy to see that for $x \in [z, 10z]$, the first term in the definition of Γ can be either positive or negative, but it is of order ηz^3 . The second term in the definition of Γ is positive and for $x \in [z, 10z]$, it is of order ηz^4 . Lastly, the third term in the definition of Γ , may be positive or negative, but in either case, it will be of order εz for $x \in [z, 10z]$. Therefore, for z sufficiently small, Γ is dominated by the second term in the definition B , i.e., by $2\varepsilon c / \bar{\sigma}^2$. Hence, the argument that was used for the linear problem in order to show that the integrand of the second term in the definition of $I_1(\varepsilon, \eta, T, t^*, \hat{x}, M)$ is in fact positive when $T - s$ is large enough, allows us to reach to the same conclusion here as well, given that z is chosen sufficiently small.

6 Appendix A

In this appendix we provide proofs of some auxiliary lemmas used in the main body of the manuscript.

Proof of Lemma 4.1. Without loss of generality, we can restrict attention to $n = 2$. We have

$$\partial_t U^\delta(t, x) = \rho_1(t, x; \delta) \partial_t \tilde{U}_1(t, x) + \rho_2(t, x; \delta) \partial_t \tilde{U}_2(t, x)$$

$$DU^\delta(t, x) = \rho_1(t, x; \delta) D\tilde{U}_1(t, x) + \rho_2(t, x; \delta) D\tilde{U}_2(t, x)$$

and

$$\begin{aligned} D^2 U^\delta(t, x) &= \frac{1}{\delta} DU^\delta(t, x)^2 - \rho_1(t, x; \delta) \left[\frac{1}{\delta} D\tilde{U}_1(t, x)^2 - D^2 \tilde{U}_1(t, x) \right] \\ &\quad - \rho_2(t, x; \delta) \left[\frac{1}{\delta} D\tilde{U}_2(t, x)^2 - D^2 \tilde{U}_2(t, x) \right]. \end{aligned}$$

Omitting function arguments for notational convenience, for $\varepsilon \leq \delta$

$$\begin{aligned}
& \partial_t U^\delta + \left[DU^\delta b - \frac{1}{2} \left| \sigma DU^\delta \right|^2 \right] + \frac{\varepsilon}{2} \alpha D^2 U^\delta \\
&= \rho_1 \partial_t \tilde{U}_1 + \rho_2 \partial_t \tilde{U}_2 + \rho_1 D \tilde{U}_1 b + \rho_2 D \tilde{U}_2 b - \frac{1}{2} \left| \sigma \left(\rho_1 D \tilde{U}_1 + \rho_2 D \tilde{U}_2 \right) \right|^2 \\
&\quad + \frac{\varepsilon}{2} \frac{1}{\delta} \left| \sigma \left(\rho_1 D \tilde{U}_1 + \rho_2 D \tilde{U}_2 \right) \right|^2 - \frac{\varepsilon}{2} \frac{1}{\delta} \left[\rho_1 \alpha (D \tilde{U}_1)^2 + \rho_2 \alpha (D \tilde{U}_2)^2 \right] \\
&\quad + \frac{\varepsilon}{2} \left[\rho_1 \alpha D^2 \tilde{U}_1 + \rho_2 \alpha D^2 \tilde{U}_2 \right] \\
&= \rho_1 \left[\partial_t \tilde{U}_1 + D \tilde{U}_1 b - \frac{1}{2} \left| \sigma D \tilde{U}_1 \right|^2 + \frac{\varepsilon}{2} \alpha D^2 \tilde{U}_1 \right] \\
&\quad + \rho_2 \left[\partial_t \tilde{U}_2 + D \tilde{U}_2 b - \frac{1}{2} \left| \sigma D \tilde{U}_2 \right|^2 + \frac{\varepsilon}{2} \alpha D^2 \tilde{U}_2 \right] \\
&\quad + \frac{1}{2} \left(1 - \frac{\varepsilon}{\delta} \right) \left[\rho_1 \left| \sigma D \tilde{U}_1 \right|^2 + \rho_2 \left| \sigma D \tilde{U}_2 \right|^2 - \left| \sigma \left(\rho_1 D \tilde{U}_1 + \rho_2 D \tilde{U}_2 \right) \right|^2 \right] \\
&\geq \frac{1}{2} \left(1 - \frac{\varepsilon}{\delta} \right) \left[\rho_1 \left| \sigma D \tilde{U}_1 \right|^2 + \rho_2 \left| \sigma D \tilde{U}_2 \right|^2 - \left| \rho_1 \sigma D \tilde{U}_1 + \rho_2 \sigma D \tilde{U}_2 \right|^2 \right] + \rho_1 \gamma_1 + \rho_2 \gamma_2 \\
&\geq \rho_1 \gamma_1 + \rho_2 \gamma_2,
\end{aligned}$$

where the last line is due to the convexity of $f(x) = x^2$. ■

Lemma 6.1 *Let $U(t, x)$ and $W(t, x)$ be two continuously differentiable functions from $[0, T] \times \mathbb{R} \mapsto \mathbb{R}$. Assume that b and σ are Lipschitz continuous. Set $\bar{u}(t, x) = -\sigma(x)DU(t, x)$, $v \in \mathcal{A}$ and let $\hat{X}^\varepsilon(s)$ solve*

$$d\hat{X}^\varepsilon(s) = -b(\hat{X}^\varepsilon(s))ds + \sigma(\hat{X}^\varepsilon(s)) \left[\sqrt{\varepsilon} dB(s) - [\bar{u}(s, \hat{X}^\varepsilon(s)) - v(s)]ds \right], \quad \hat{X}^\varepsilon(0) = y.$$

Then for every $\varepsilon > 0$, $v \in \mathcal{A}$ and stopping time $\hat{\tau}^\varepsilon \leq T$, we have, with probability 1,

$$\begin{aligned}
& \int_0^{\hat{\tau}^\varepsilon} \left[\frac{1}{2} v(s)^2 - \bar{u}(s, \hat{X}^\varepsilon(s))^2 \right] ds \\
& \geq 2W(0, y) - 2W(\hat{\tau}^\varepsilon, \hat{X}^\varepsilon(\hat{\tau}^\varepsilon)) + 2\sqrt{\varepsilon} \int_0^{\hat{\tau}^\varepsilon} DW(s, \hat{X}^\varepsilon(s)) \sigma(\hat{X}^\varepsilon(s)) dB(s) \\
& \quad + 2 \int_0^{\hat{\tau}^\varepsilon} \mathcal{G}^\varepsilon[W](s, \hat{X}^\varepsilon(s)) ds - \int_0^{\hat{\tau}^\varepsilon} \left| \sigma(\hat{X}^\varepsilon(s)) \left(DW(s, \hat{X}^\varepsilon(s)) - DU(s, \hat{X}^\varepsilon(s)) \right) \right|^2 ds
\end{aligned}$$

Proof. We make use of the min/max representation

$$\mathbb{H}(x, p) = \inf_v \sup_u \left[p(b(x) - \sigma(x)u + \sigma(x)v) - \frac{1}{2}u^2 + \frac{1}{4}v^2 \right].$$

Assume we use the control $\bar{u}(t, x) = -\sigma(x)DU(t, x)$ for the design of the scheme and choose $p = DW(t, x)$. Then

$$\begin{aligned}
& \inf_v \left[DW(t, x)(b(x) - \sigma(x)\bar{u}(x) + \sigma(x)v) - \frac{1}{2}\bar{u}(t, x)^2 + \frac{1}{4}v^2 \right] \\
&= DW(t, x)(b(x) + \sigma^2(x)DU(t, x) - 2\sigma^2(x)DW(t, x) \\
&\quad - \frac{1}{2}|\sigma(x)DU(t, x)|^2 + |\sigma(x)DW(t, x)|^2 \\
&= DW(t, x)b(x) - \frac{1}{2}|\sigma(x)DW(t, x)|^2 - \frac{1}{2}|\sigma(x)(DW(t, x) - DU(t, x))|^2 \\
&= \mathbb{H}(x, DW(t, x)) - \frac{1}{2}|\sigma(x)(DW(t, x) - DU(t, x))|^2.
\end{aligned}$$

Applying Itô's formula to $W(s, \hat{X}^\varepsilon(s))$ then gives

$$\begin{aligned}
W(\hat{\tau}^\varepsilon, \hat{X}^\varepsilon(\hat{\tau}^\varepsilon)) &= W(0, y) + \int_0^{\hat{\tau}^\varepsilon} \left[\partial_s W(s, \hat{X}^\varepsilon(s)) + \right. \\
&\quad \left. + DW(s, \hat{X}^\varepsilon(s)) \left[b(\hat{X}^\varepsilon(s)) - \sigma(\hat{X}^\varepsilon(s))\bar{u}(s, \hat{X}^\varepsilon(s)) + \sigma(\hat{X}^\varepsilon(s))v(s) \right] \right] ds \\
&\quad + \int_0^{\hat{\tau}^\varepsilon} \frac{\varepsilon}{2} \sigma^2(\hat{X}^\varepsilon(s)) D^2 W(s, \hat{X}^\varepsilon(s)) ds + \int_0^{\hat{\tau}^\varepsilon} \sqrt{\varepsilon} DW(s, \hat{X}^\varepsilon(s)) \sigma(\hat{X}^\varepsilon(s)) dB(s) \\
&\geq \int_0^{\hat{\tau}^\varepsilon} \left[\frac{1}{2} \bar{u}(s, \hat{X}^\varepsilon(s))^2 - \frac{1}{4} v(s)^2 \right] ds + \sqrt{\varepsilon} \int_0^{\hat{\tau}^\varepsilon} DW(\hat{X}^\varepsilon(s)) \sigma(\hat{X}^\varepsilon(s)) dB(s) \\
&\quad + \int_0^{\hat{\tau}^\varepsilon} \left[\partial_s W(s, \hat{X}^\varepsilon(s)) + \mathbb{H}(\hat{X}^\varepsilon(s), DW(s, \hat{X}^\varepsilon(s))) + \frac{\varepsilon}{2} \sigma^2(\hat{X}^\varepsilon(s)) D^2 W(s, \hat{X}^\varepsilon(s)) \right] ds \\
&\quad - \frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} \left| \sigma(\hat{X}^\varepsilon(s)) \left(DW(s, \hat{X}^\varepsilon(s)) - DU(s, \hat{X}^\varepsilon(s)) \right) \right|^2 ds \\
&= \int_0^{\hat{\tau}^\varepsilon} \left[\frac{1}{2} \bar{u}(\hat{X}^\varepsilon(s))^2 - \frac{1}{4} v(s)^2 \right] ds + \sqrt{\varepsilon} \int_0^{\hat{\tau}^\varepsilon} DW(s, \hat{X}^\varepsilon(s)) \sigma(\hat{X}^\varepsilon(s)) dB(s) \\
&\quad + \int_0^{\hat{\tau}^\varepsilon} \mathcal{G}^\varepsilon[W](s, \hat{X}^\varepsilon(s)) ds - \frac{1}{2} \int_0^{\hat{\tau}^\varepsilon} \left| \sigma(\hat{X}^\varepsilon(s)) \left(DW(s, \hat{X}^\varepsilon(s)) - DU(s, \hat{X}^\varepsilon(s)) \right) \right|^2 ds
\end{aligned}$$

Rearranging this expression concludes the proof of the lemma. ■

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